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How Much Do Wave Length Shifters Used for Čerenkov Counters Scintillate?

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Summary. — The increase in light output of liquid Čerenkov counters obtained by addition of wave length shifting substances (as suggested by many authors) has been studied. It is shown that most of the observed increase is due to scintillation and not to actual wave length shifting. Crystalline sodium salycilate coating of the counter walls has also been used for the same purpose. Its average quantum efficiency is found to be relatively high but its practical use is limited by the fact that its reflectivity in the visible is lower than that of MgO.

1. – Introduction.

While constructing undirectional Čerenkov counters of different types, we got interested in wave length shifters. The task of a wave length shifter would be to absorb the Čerenkov light in the ultra-violet and to reemit it in the wave length region (3500 ÷ 5500 Å) in which the photomultiplier is sensitive. If, for example, it were possible to convert all the Čerenkov photons between 2000 Å and 3500 Å into visible photons, the increase in the number of usable photons would be by a factor of about three. This technique would therefore improve, for example, the resolution of thin Čerenkov counters, a type whose applicability is essentially limited by poor photomultiplier statistics.

Several wave length shifting materials have been used or suggested for this purpose [1-4]. In particular it has been claimed that dissolution of small

[1] E. HEIBERG and J. MARSHALL: *Rev. Scient. Instr.*, **27**, 618 (1956).
[2] N. PORTER: *Nuovo Cimento*, **5**, 526 (1957).
[3] W. B. JONES, H. R. KRATZ and J. ROUVINA: *Rev. Scient. Instr.*, **28**, 167 (1957).
[4] K. SAITO and K. SUGA: *Nuovo Cimento*, **11**, 600 (1959).

quantities of beta methyl umbelliferone in water ([2]), and of POPOP in carbon tetrachloride ([3]), increases the Čerenkov light output by a factor of about two. Very recently, increases by factors as high as 4 have been reported, by using amino G acid ([4]).

As many of these materials are also known to be weak scintillators, it is necessary to make sure that the observed increase in light is not due to scintillations. This is generally tested by measuring separately the light output obtained with particles well below the Čerenkow threshold and assuming scintillation light to be proportional to the energy loss. When, however, the energy of the particles is very low, possible saturation effects make the comparison not clear cut.

2. – Experimental procedure and results.

2·1. *Wave length shifters in solution.* – The method employed to ensure adequate separation of wave length shifting and scintillation was to prevent the ionizing particles from penetrating into the wave length shifting solution. To this purpose we constructed a Čerenkov light source, consisting of a small cell which contained a 20 millicurie ^{89}Sr beta source ($E_{max} = 1.45$ MeV;

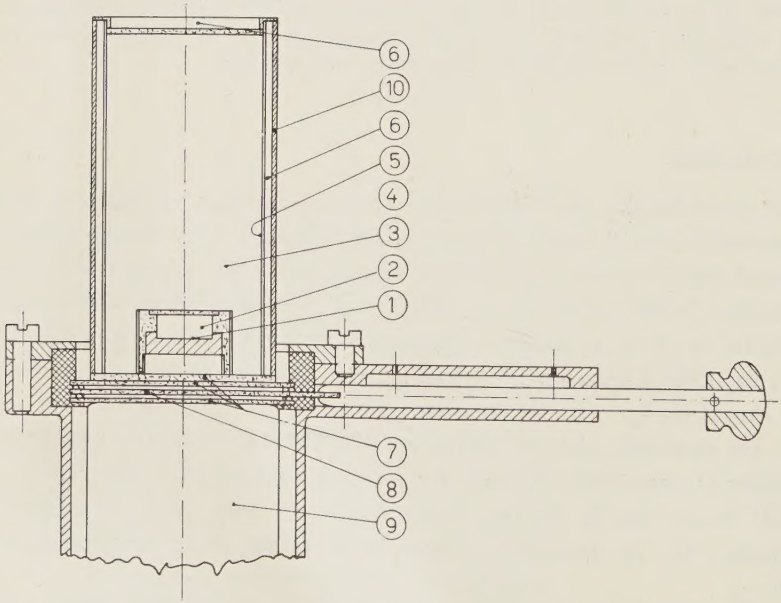

Fig. 1. – 1) Radioactive layer (Sr 89). 2) Water. 3) Quartz window. 4) Wave shifting solution. 5) Plexiglass (0.5 mm). 6) Diffusing layer (MgO). 7) Glass windows (Chance OW 1). 8) Shutter. 9) Photomultiplier (R.C.A. 6342). 10) Aluminium.

$R_{max} = 6$ mm in H_2O) covered with 6 mm of water and sealed with a thin quartz window.

This Čerenkov light source was immersed in a vessel having white diffusing walls, filled with the liquid under study and viewed by a R.C.A. 6342 photomultiplier (Fig. 1). The transmittance of the solutions used was checked by means of a quartz spectrophotometer (Fig. 2).

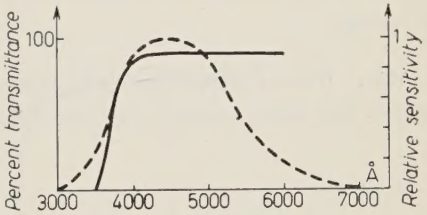

Fig. 2. – Transmittance of solutions. - - - 0.03 g/l β methyl umbelliferone in H_2O measured against H_2O. ——— POPOP saturated in CCl_4 measured against CCl_4. Cell length = 1 cm.

We measured the photomultiplier anode current at different concentrations of the solute. To reduce background and ensure reproducibility of the measurements, the photomultiplier was selected for low dark current, its cathode was kept at ground potential and provision was taken for changing the liquid without removing the anode high voltage. With these precautions, the reproducibility of the measurements was better than $\pm 2\%$. We then removed the quartz window, so that the β-rays did penetrate the solution, and repeated the measurement.

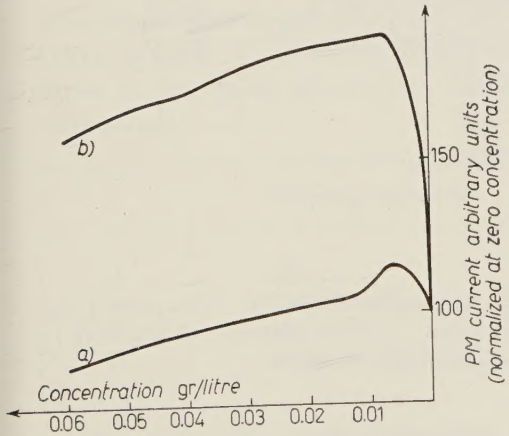

Fig. 3. – β-methyl umbelliferone in H_2O. (*a*) Solution shielded from β-source. (*b*) Solution in contact with β-source.

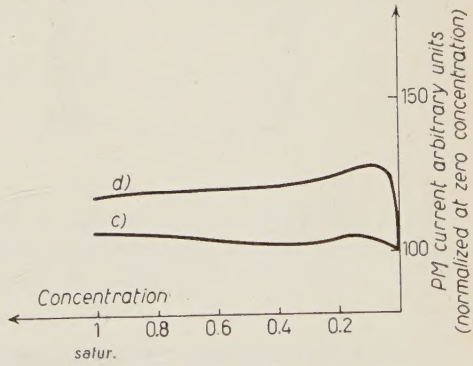

Fig. 4. – POPOP in CCl_4 (*c*) Solution shielded from β source. (*d*) Solution in contact with β source.

The results are given in Fig. 3 and 4. The measured currents have been normalized to zero concentration, to account for the somewhat different optics. We note that while the increase of light due to actual wave length shifting is rather low, (curves *a*) and *c*)) the contribution from scintillations in the solutions (curve *b*) and *d*)) can certainly not be neglected. This is in contrast

with the observations of PORTER (for beta methil umbelliferone in water) and of JONES *et al.* (for POPOP in CCl_4) (5).

We conclude therefore that the addition of the above solutes to a liquid Čerenkov counter will destroy not only the directional properties, but also the velocity dependence of its light intensity, including the characteristic threshold property.

2·2. *Wave length shifting by crystalline sodium salycilate.* – Another material which has been extensively used as wave length shifter is crystalline sodium

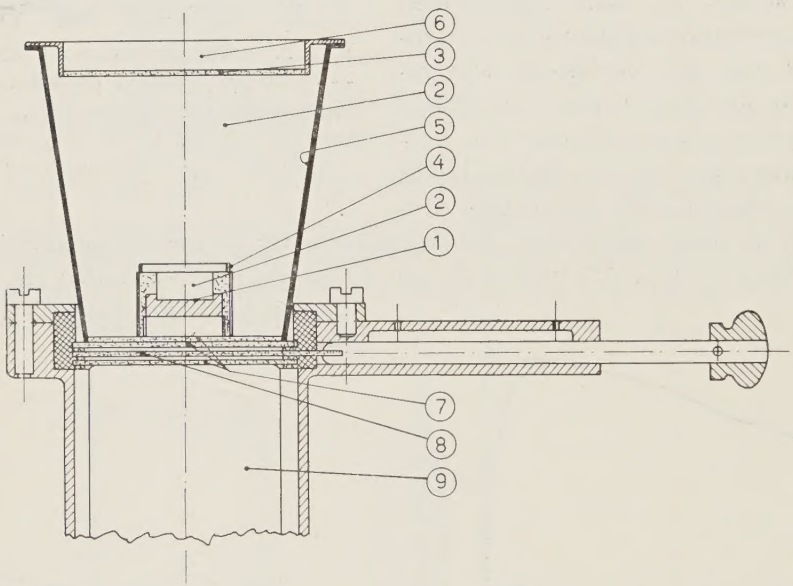

Fig. 5. – Radioactive layer (Sr 89). 2) Water. 3) Quartz window. 4) Ultraviolet Filter (Chance OY 10). 5) Black point. 6) Diffusing layer. 7) Glass windows (Chance OW 1). 8) Shutter. 9) Photomultiplier (R.C.A. 6342).

(5) A more quantitative comparison is rather difficult, because of the different geometrical and optical arrangements, the different spectral response of the photomultipliers used, and the different energy spectrum of the incident particles. The latter effect can be roughly corrected for using published data on the beta spectrum of ^{89}Sr. For a minimum ionizing particle traversing our vessel, for example, the increase in light output of a solution of beta methyl umbelliferone at optimum concentration relative to pure water would be reduced to 47% (of this, 32% would be due to scintillation, and the rest to actual wave length shifting). The difference between this figure and the 100% increase observed by Porter can probably be explained in terms of the remaining above mentioned experimental differences. Similar considerations hold for the case of carbon tetrachloride, where the geometrical differences are even more important, due to the strong selective absorption of the medium.

salycilate ([6-10]). A quantum efficiency of about 50% has been reported ([11]). To test its practical efficiency in the case of Čerenkov radiation we used the arrangement shown in Fig. 5. By measuring the photomultiplier current as before, we compared the light diffused into the photo multiplier by a thick layer of sodium salycilate powder with that diffused by a thick layer of MgO.

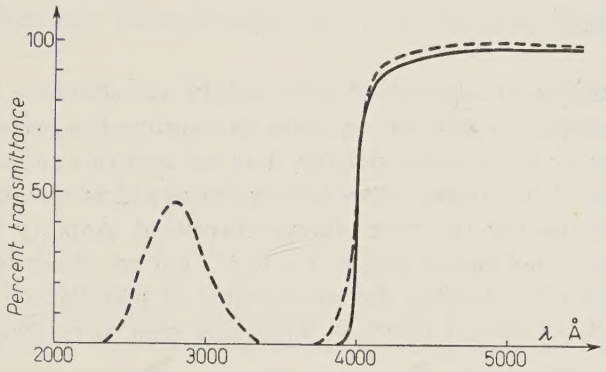

Fig. 6. – —— Trasmittance of ultraviolet filter Chance OY 10 measured against air. – – – Spectral sensitivity of photomultiplier (S 11).

The measurements were repeated after inserting an ultra-violet filter (chance OY10, see Fig. 6) above the Čerenkov light source, as shown. The results are given in the Table (a constant background of $4.3 \cdot 10^{-9}$ A has been already subtracted):

	Diffusing material	Filter	Current (10^{-9} A)
A	MgO	No	31.5 ± 0.5
B	Sodium salycilate	No	35.5 ± 0.5
C	MgO	Yes	29.5 ± 0.4
D	Sodium salycilate	Yes	23.5 ± 0.5

The ratio $D/C = 0.80$ shows that the reflectance of sodium salycilate in the visible is smaller than that of MgO, whereas the relatively large ratio $B/D = 1.50$ obtained with sodium salycilate (as compared with $A/C = 1.07$ for

([6]) G. Dejardin and R. Schwegler: *Rev. Optique*, **13**, 313 (1934).
([7]) F. S. Johnson, K. Watanabe and R. Tousey: *Journ. Opt. Soc. Am.*, **41**, 702 (1951).
([8]) E. Inn: *Spectrochimica Acta*, **2**, 2 (1955).
([9]) K. Watanabe and E. Inn: *Journ. Opt. Soc. Am.*, **43**, 33 (1953).
([10]) D. H. Thurnau: *Journ. Opt. Soc. Am.*, **46**, 246 (1956).
([11]) C. O. Muelhouse: *Nucleonics*, **14** (4), 38 (1956).

MgO), indicates that wave length shifting from the ultra-violet into the visible actually occurs. The effective average quantum efficiency, if estimated over the range $(2000 \div 3500)$ Å, turns out to be about 20%. For practical purposes, however, the gain is given by the ratio B/A, which is only 1.12.

3. – Conclusions.

a) The suggested addition of beta methyl umbelliferone to water and of POPOP to carbon tetrachloride in order to improve the performance of Čerenkov counters by wave length shifting, does not lead to a substantial increase of the Čerenkov light output. The increase observed is mainly due to scintillations which destroy the characteristic threshold property of a Čerenkov counter. We have not tested amino G acid ([4]), but we think that great care must be applied in evaluating the contribution of scintillation from a measurement of the light output obtained with very slow particles.

b) Crystalline sodium salycilate has a greater efficiency in shifting Čerenkov light from the ultra violet into the visible. Its practical use as a coating for the counters walls is limited, however, by the fact that its reflectance is inferior to that of MgO.

RIASSUNTO

Alcuni autori hanno suggerito l'impiego di convertitori di frequenza (wave shifters) in soluzione per aumentare la resa in luce di contatori di Čerenkov liquidi. Si dimostra che, almeno in due casi, l'aumento osservato in rendimento luminoso è dovuto in gran parte a scintillazione diretta della soluzione, piuttosto che ad effettiva conversione di frequenza della luce di Čerenkov. È stato studiato anche il rendimento di conversione di uno strato di salicilato di sodio in polvere, opportunamente depositato sulle pareti del contenitore: il rendimento quantistico medio per la regione ultravioletta vicina dello spettro di Čerenkov risulta relativamente elevato, ma il guadagno praticamente ottenibile è limitato dal fatto che il potere riflettente del salicilato di sodio nel visibile è inferiore a quello dell'ossido di magnesio.

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The Nuclear Compressibility from Isotope Shift Data.

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Summary. — A general theory of isotope shift in heavy atoms taking into account the nuclear specific effects as well as those due to electronic shells is given. Only the « odd staggering » effect is not considered owing to the particular choosing of nuclei (even-even). The problem is resolved rigorously under the assumption that the proton distribution in nuclei is that of Fermi. For the computation of compressibility the model of Wilets with some modification is used. The discussion of results for a wide range of mass numbers—including the high nuclear deformation region—leads to the need of Johnson-Teller effect and to a value for compressibility of about 77 MeV.

1. – Introduction.

The correct solving of the isotope shift problem in atomic spectra encounters the essential difficulty of obtaining the Darwin-Gordon differential system solutions for a realistic charge distribution—this is practically impossible. In this work we show that this problem is not a necessary one, because it can be exactly reduced to the equivalent case of a homogenous charged nucleus. The solutions for this last case were given by us elsewere [1].

The corrections linked to electrons shielding from inner shells by the optical electron during the transition were estimated on the Crawford and Schawlow model [2] for a set of elements from Na to Fr. They allow us to make a suitable interpolation for other elements within the range.

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[1] N. J. Ionesco-Pallas: *Phys. Rev.* (1959), submitted for publication.

[2] M. F. Crawford and A. L. Schawlow: *Phys. Rev.*, **76**, 1310 (1949).

For the region of high nuclear deformations which extends up to $\sim N = 126$, the abnormal large isotope shifts are satisfactorily explained by the Wilets and Breit theory ([3,4]).

After some modifications of Wilets theory on nuclear compressibility, we tried to compute this amount using about 18 elements in the range of $Z = 40 \div 80$, in both following hypotheses:

a) equality between the nuclear radii due to charge and mass distributions;

b) inequality between these radii.

The last hypothesis leads to a value of about 77 MeV for the compressibility per nucleon which—ignoring the surface effects—seems to correspond to reality. The above value is in agreement with the theoretical calculations of FEENBERG ([5]), BRUCKNER ([6]) and WILETS ([7]).

2. – Spherical incompressible nuclei.

Let us suppose, firstly, that we have succeded in reducing the problem of electromagnetic interaction of a real nucleus to an equivalent one in which the nucleus is homogenously charged. In this case, the Broch perturbation theory of boundary conditions can be applied in the following circumstances:

a) The approximate Breit wave-functions for the electron outside the nucleus are used.

b) The Fermi-Segré formula for the density of unrelativistic probability of electron is considered.

c) Some approximations are made, suggested by the inequality $y_0 = (2ZR/a_B) \ll 1$. This leads to a formula which gives the isotope shifts due to an electron with k Dirac's quantum number

$$\frac{\delta E}{hc} \approx 2\pi R_\infty \cdot \frac{a_H^3}{Z_i} \cdot \psi_n^2(0) \cdot \frac{2\varrho}{\Gamma^2(1+2\varrho)} \cdot \frac{1+(k-\varrho)\zeta_k}{1+(k+\varrho)\zeta_k} y_0^{2\varrho}, \tag{2.1}$$

([3]) L. WILETS, D. L. HILL and K. W. FORD: *Phys. Rev.*, **91**, 1488 (1953).
([4]) G. BREIT: *Rev. Mod. Phys.*, **30**, 507 (1958).
([5]) E. FEENBERG: *Phys. Rev.*, **59**, 149 (1941).
([6]) K. A. BRUECKNER: *Phys. Rev.*, **97**, 1353 (1955).
([7]) L. WILETS: *Phys. Rev.*, **101**, 1805 (1956).

where $\zeta_k = -F(R)/aG(R)$ rest to be derived from the solutions of Darwin-Gordon differential system. In the particular case of a $s_{\frac{1}{2}}$ electron and in the approximation $-U(R)/m_0c^2 \gg 1$ we obtain

$$\zeta_{-1} = \tfrac{2}{5}\{1 + 0.106\,233\,a^2 + 0.016\,649\,a^4 + 0.002\,788\,a^6 + 0.000\,479\,a^8 + \ldots\}\,. \tag{2.2}$$

Here

$$a = \frac{Ze^2}{\hbar c}\,; \qquad \varrho = \sqrt{k^2 - a^2}\,.$$

Sommerfeld's treatment of the problem leads just to the same result as those given in (2.1) [8].

Let us admit also that the true charge distribution does not differ much from a constant one, in this case the (2.1) formula is modified so that only the ζ value is altered.

Applying now the Breit perturbations theory [9], this gives in its turn

$$\frac{\delta E}{hc} \approx 2\pi R_\infty \frac{a_H^3}{Z_i}\psi_n^2(0)\,\frac{2k(k-\varrho)}{\Gamma^2(1+2\varrho)}\,y_0^{2\varrho}\int_0^\infty [1 - xf(x)]\,x^{2\varrho-1}\,\mathrm{d}x\,, \tag{2.3}$$

where

$$x = r/R\,; \qquad f(x) = +U(r)/U(R)\,.$$

In the limiting case $\varrho \to 1$ the formulae (2.1) and (2.3) must coincide with each other, and this fact gives us the general relationship between ζ and charge distribution $g(x)$

$$\zeta_{-|k|}(0) = \frac{1}{2|k|}\left\{1 - \frac{1}{(2|k|+1)}\cdot\frac{\int_0^\infty x^{2(|k|+1)}\,g(x)\,\mathrm{d}x}{\int_0^\infty x^2 g(x)\,\mathrm{d}x}\right\}. \tag{2.4}$$

For the states with a given k the use of (2.4) requires another relation which can be established by the same method of extreme correspondence,

$$\zeta_{-|k|}\cdot\zeta_{+|k|} + \frac{1}{a^2} \approx 1\,. \tag{2.5}$$

We now adopt as realistic charge distribution in nuclei the Fermi function for nucleons distribution, together with the assumption that between neutron

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[8] N. J. Ionesco-Pallas: *Prog. Theor. Phys.*, **21**, no. 4 (1959).

[9] G. Breit and J. E. Rosenthal: *Phys. Rev.*, **41**, 459 (1932).

and proton distributions there is no difference [10]

$$g(x) = g(0)\,\frac{1+\exp\left[-\dfrac{r_0}{a}\right]}{1+\exp\left[+\dfrac{r-r_0}{a}\right]} \tag{2.6}$$

Here r_0 is the inflexion radius, $a = 0.575\cdot 10^{-13}$ cm and $\{3/4\pi g(0)\}^{\frac{1}{3}} = p = 1.121\cdot 10^{-13}$ cm. The analytical form for ζ can now be derived with help of the (2.6) distribution and of the mathematical identity

$$\int_0^\infty \frac{x^{2q}\,\mathrm{d}x}{1+\exp\,[x-x_0]} \equiv \frac{x_0^{2q+1}}{(2q+1)} + 2(2q)!\sum_1^q{}_\mu \left(1-\frac{1}{2^{2\mu-1}}\right)\frac{\zeta(2\mu)x_0^{2q-2\mu+1}}{(2q-2\mu+1)!} + \tag{2.7}$$

$$+(2q)!\sum_1^\infty{}_\mu \frac{(-1)^{\mu-1}}{\mu^{2q+1}}\exp\,[-\mu x_0]\,, \quad (q \text{ integer}),$$

(here ζ is the Riemann's function). Hence

$$\zeta_{-|k|}(0) \approx \frac{1}{2\,|k|}\left\{1-\frac{3}{(2\,|k|+1)(2\,|k|+3)}\left(\frac{r_0}{R}\right)^{2|k|}\cdot\right. \tag{2.8}$$

$$\left.\cdot\left[1+\frac{\pi^2}{3}\,|k|\,(2\,|k|+5)\left(\frac{a}{r_0}\right)^2+\ldots\right]\right\}.$$

For the inflexion and mean square radii, we have

$$r_0 \sim pA^{\frac{1}{3}} - \frac{\pi^2}{3}\,\frac{a^2}{p}\,A^{-\frac{1}{3}} + \frac{\pi^6}{81}\,\frac{a^6}{p^5}\,A^{-5/3} - \ldots\,, \tag{2.9}$$

$$R \sim \sqrt{\frac{5}{3}}\,(\overline{r^2})^{\frac{1}{2}} \sim pA^{\frac{1}{3}} + \frac{5}{6}\pi^2\,\frac{a^2}{p}\,A^{-\frac{1}{3}} - \frac{7}{8}\pi^4\,\frac{a^4}{p^3}\,A^{-1} + \ldots\,. \tag{2.10}$$

Introducing in (2.8) the two kinds of radii from (2.9) and (2.10) we obtain for $\zeta_{-1}(0)$ a value of $\frac{2}{5}$ which means that the problem of reduction to an equivalent case discussed above is exactly possible if we choose as nuclear radius the mean square radius. We shall show further that this condition is not a restrictive one.

Let us, therefore, compute the « energetic radius » of a heavy nucleus by mean of formula

$$\frac{1}{2}\,e^2\int_{(\tau)}\int_{(\tau')}\frac{g(\boldsymbol{r})\cdot g(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\tau\,\mathrm{d}\tau' = \frac{3}{5}\,\frac{Z^2e^2}{R_e}\,, \tag{2.11}$$

[10] L. R. B. Elton: *Rev. Mod. Phys.*, **30**, 557 (1958).

where $g(r)$ is the proton density linked to the nucleon density by the relation $g_Z(\boldsymbol{r}) = (Z/A)g_A(\boldsymbol{r})$ and $g_A(\boldsymbol{r})$ is given by (2.6). Taking into account the spherical symetry of nucleus, we obtain

$$R_e = \frac{2}{15}\frac{p^6}{a^5}A^2S^{-1}\,; \qquad S \sim \frac{2}{15}x_0^5 + \frac{\pi^2}{9}x_0^3 + 0(x_0^2)\,,$$

where $x_0 = r_0/a$. Hence

$$R_e \sim pA^{\frac{1}{3}} + \frac{5}{6}\pi^2\frac{a^2}{p}A^{-\frac{1}{3}} + 0(A^{-1}) - \ldots\,. \tag{2.12}$$

The reduction of the problem to the equivalent case of a homogeneous nucleus is also consistent with the energetic definition of radius.

3. – The compressibility of heavy nuclei.

If we perform the derivative of (2.1) with respect to the nuclear radius R we obtain a proportionality between relative isotope shifts and $\delta R/R$. If moreover we use the $\frac{1}{3}$ power law for radius

$$R \sim r_0 A^{\frac{1}{3}}\,, \tag{3.1}$$

with $r_0 \sim 1.216 \cdot 10^{-13}$ cm as resulting from Weizsäcker mass formula we have a proportionality between the mentioned quantity and $\delta A/A$. This is the case of hard nuclei—the lack of elasticity being due to the neglect of Coulombian interactions. In reality, the variations of nuclear radius for the addition of one neutron or one proton are different. Therefore we shall write

$$\frac{\delta R}{R} \approx \frac{3A}{R}\left(\frac{\partial R}{\partial N}\right)_Z\frac{\delta A}{3A}\,. \tag{3.2}$$

For the calculation of partial derivative in (3.2) we consider the radius expression as resulting from Feenberg theory [5]

$$R = R_0(N, Z)\left[1 - \frac{E_{0c}}{E_0''}\right]^{-1}\,, \tag{3.3}$$

where E_{0c} is the Coulombian energy of an incompressible nucleus

$$E_{0c} \sim \frac{3}{5}\frac{e^2}{r_0}\frac{Z^2}{A^{\frac{1}{3}}} \tag{3.4}$$

and E_0'' is the volume compressibility

$$E_0'' \sim K_v A \,. \tag{3.5}$$

We stress that the r_0 parameter which enters in the (3.1) and (3.4) formulae may be deduced from (2.10) or (2.12) by a least square method,

$$r_0 \sim p + \frac{25}{18}\pi^2 \frac{a^2}{p} A_{\max}^{-\frac{2}{3}} \,. \tag{3.6}$$

Such, a value of about $1.226 \cdot 10^{-13}$ cm is obtained, in good agreement with that above, if we set $A_{\max} \sim 238$ (Uranium mass).

We now admit that

$$R_0(N, Z) = R_0(N + Z) \tag{3.7}$$

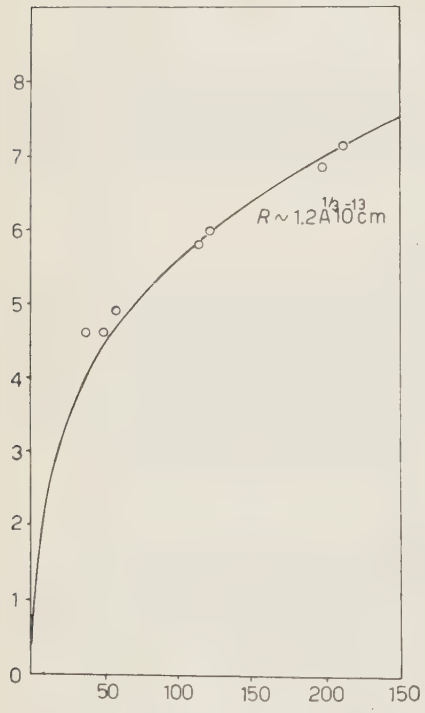

Fig. 1. – Nuclear radii plotted *versus* mass number A. The circles represent the experimental data of HAHN *et al.* The radii are measured in 10^{-13} cm (fermis). For the full line, $r_0 \sim 1.216 \cdot 10^{-13}$ cm was assumed.

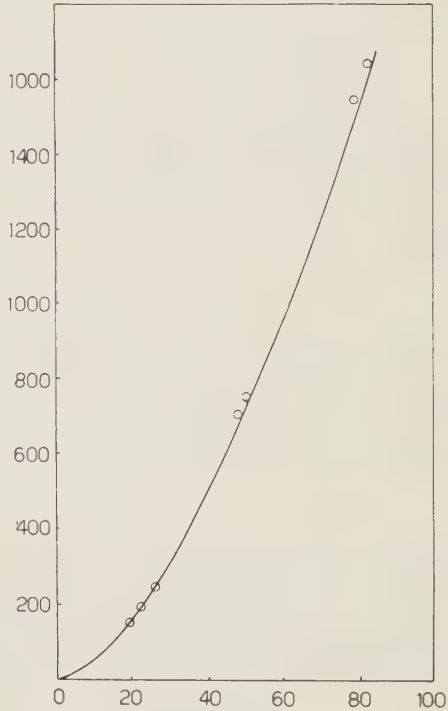

Fig. 2. – Coulombian energy of nuclei plotted as a function of charge number Z. m_0c^2 is chosen as unity. The Fermi mass distribution was used assuming a constant ratio between proton and neutron densities. The circles are the experimental data of HAHN *et al.*

which is natural for the most stable nuclei, defined by relation

$$Z_A = A[1.983 + 0.015 A^{\frac{2}{3}}]^{-1} . \tag{3.8}$$

For them, the formulae (3.3) and (3.1) must coincide. We obtain in this way a complete analytical expression of nuclear radius. The using of this expression leads to the following relationship of interest for us,

$$Q = \frac{3A}{R}\left(\frac{\partial R}{\partial N}\right)_Z \approx 1 - 6\left[1 - \frac{Z}{A^{\frac{1}{3}}}10^{-2}\right]\frac{E_{0c}}{E_0'' - E_{0c}} . \tag{3.9}$$

The reason for the derivation of this formula is in fact the same as in Wilets' paper ([7]), but he did not succeed in giving an explicite form. Moreover, those results are to some extent erroneous owing to the incorrect Coulombian energies. Recent measurements ([11]) confirme the values adopted by us for both nuclear radii and Coulombian energies.

4. – Screening of inner electrons during the transition.

As resulting in Crawford and Schawlow theory, during the transition, the optical electron penetrates the inner shells and diminishes the nuclear charge for $s_{\frac{1}{2}}$ electrons. Our calculations made in the same circumstances as the mentioned authors (the Thomas-Fermi statistical model) for many elements of the periodic table such as alkalis, inert gases, Sn, Hg, Pb, lead to the following conclusions about the shielding coefficient.

1) The screening increases with the nuclear charge Z, roughly as Z^2.

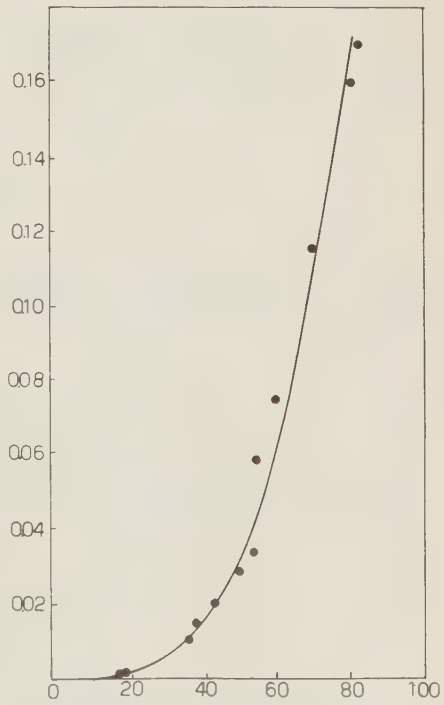

Fig. 3. – Screening of electrons from inner shells by valence electron during transition, *versus* charge number Z. The full circles are computed supposing a standard Thomas-Fermi model, the dark line represents a least squares fit with a polynomial expression. We stress the falls at charge number corresponding to electronic closed shells.

([11]) B. Hahn, D. G. Ravenhall and R. Hofstadter: *Phys. Rev.*, **101**, 1131 (1956).

2) Sudden jumps are observed when the electronic shells are thoroughly occupied.

3) The shielding correction is sensitive only for heavy atoms and its order of magnitude is given by the last closed shell. The full line plotted in Fig. 3 represents the best fit by the least squares ajustments.

The shielding corrections computed as above are to some extent overestimated.

5. – Deformed and excited nuclei.

For the nuclei in the range of high deformations, another correction which should take into account the oblateness or the prolateness must be introduced as well. The numerical calculations which we have made for Sn, Sm, Eu and Gd based upon Kopferman, Wilets and Breit's formula ([12,3,4])

$$\frac{\delta E_\alpha}{\delta E_v} \sim \frac{3}{10}(2\varrho+3)A\,\delta\alpha^2\left\{1+\frac{1}{7}(2\varrho+3)\bar{\alpha}+\ldots\right\} \tag{5.1}$$

agree well wit the observed shifts. Here $\delta\alpha^2$ is the contribution to whole isotope shift due to isometric nuclear deformation, and α is the deformation parameter tied to intrinsic quadripole moment by the relation

$$\alpha \sim \frac{5}{6}\frac{Q_0}{ZR_0^2}\,. \tag{5.2}$$

The intrinsec quadripole is in its turn linked to an observable moment by the well known relationship from strong coupling theory of A. Bohr ([13])

$$Q_0 \sim \frac{I+1}{I}\frac{I+\frac{3}{2}}{I-\frac{1}{2}}Q\,. \tag{5.3}$$

In order to express $\delta\alpha^2$ in terms of changes of other quantities tied more directly to nuclear structure, we also stress the formula

$$\frac{Q_2}{Q_1} \sim \frac{I_2-\frac{1}{2}}{I_1-\frac{1}{2}}\frac{I_1+1}{I_2+1}\left(\frac{\overline{r_2^2}}{\overline{r_1^2}}\right). \tag{5.4}$$

([12]) P. Brix and H. Kopfermann: *Zeits. f. Phys.*, **126**, 344 (1949).
([13]) A. Bohr and B. R. Mottelson: *Dan. Mat.-Fys. Medd.*, **30**, no. 1 (1955).

Combining the equations (5.2), (5.3) and (5.4) we obtain

$$\frac{\delta\alpha^2}{\alpha^2} \sim -3\,\frac{I_1 I_2 + \frac{3}{4}(I_1 + I_2)}{(I_1 I_2 + \frac{3}{2} I_2)^2}\,\delta I + 2\,\frac{\delta(\overline{r^2})}{(\overline{r^2})}\,. \tag{5.5}$$

From the eq. (5.5) we may readily draw the following interesting conclusion: sofar as δE_α is positive the deformation variation must not lead to a spin change. This is obvious on the basis of weak change of mean square radius. The proper case is that of Eu. Inversely, a sudden variation of spin must lead to a negative δE_α but this is not customary without a state being excited. The only example of this kind was recently reported by MELISSINOS and DAVIS [14] who had found $\delta E({}^{198}_{80}\mathrm{Hg} - {}^{197}_{80}\mathrm{Hg}^*(i\ 13/2)) < \delta E({}^{198}_{80}\mathrm{Hg} - {}^{197}_{80}\mathrm{Hg}(p\ \tfrac{1}{2}))$. We try below to explain this phenomenon on the basis of formula (5.5). Using the (5.4) relation between the quadripole momenta of ^{201}Hg and ^{197}Hg nuclei and supposing that $^{197}\mathrm{Hg}(p\ \frac{1}{2})$ satisfy (3.1) we have (in barns)

$$1 + (\mu_e - 1) \sim \left(1 - \frac{8}{3A}\right)\left(1 + \frac{\delta\overline{r^2}}{\overline{r^2}}\right) \tag{5.6}$$

or, with the help of (5.5)

$$\frac{\delta\alpha^2}{\alpha^2} \sim -3\,\frac{I_1 I_2 + \frac{3}{4}(I_1 + I_2)}{(I_1 I_2 + \frac{3}{2} I_2)^2}\,\delta I + 2\left\{\left(1 + \frac{8}{3A}\right)[1 + (\mu_e - 1)] - 1\right\} \sim \tag{5.7}$$

$$\sim -0.879 + 2.027(\mu_e - 1)\,.$$

But (5.7) can be written also in the alternative form

$$\alpha^2 \sim \alpha^{*2}[0.121 + 2.027(\mu_e - 1)] \tag{5.8}$$

which allows us to derive the α^2 for $^{197}\mathrm{Hg}(p_{\frac{1}{2}})$ in an exhausting way (we recall that this quantity cannot be evaluated directly, because of the indeterminacy which occurs)

$$\alpha^2 \sim 4.34 \cdot 10^{-4}\,\frac{[1 + (\mu_e - 1)]^2}{[0.121 + 2.027(\mu_e - 1)]} = 4.34 \cdot 10^{-4} f(\mu_e)\,. \tag{5.9}$$

From the experimental value of ratio (5.1) we obtain finally

$$\frac{\delta E_\alpha}{\delta E_v} \sim 0.34 \pm 0.076 \quad \text{and} \quad f(\mu_e) \sim 1.969 \pm 0.641\,.$$

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[14] A. MELISSINOS and S. P. DAVIS: *The Dipole and Quadripole Momenta of Excited* ^{197}Hg *Nucleus*, Mass. Inst. Tech. Depart. of Phys. and Research Labor. of Electronics (1959) (unpublished).

Now, the experimental value of μ_e was obtained from the analysis of hyperfine structure data, ($\mu_e \sim (1.5 \pm 0.3)$ barns. Introducing this quantity as argument in $f(\mu_e)$ a second value of this function results

$$f(\mu_e) \sim 2.296 \pm 0.438 .$$

This fall in good agreement with the previous value, confirming our hypotheses.

6. – The estimation of nuclear compressibility.

We dispose of a number of about 18 nuclei ([15]) from which we may try a systematization versus charge number Z. This is the most convenient for us ([16]). The conversion from A to Z scale of diverse expressions was made on the eq. (3.8). All the cases lie consequently under the line $\delta E_{\text{exp}}/\delta E^{\text{inc}}_{\text{th}} \sim 1$ Firstly, we have accepted equality between nucleon and proton distribution radii $R_t \sim R_z$, in which case the best fit with esperimental data is realized for $K_v \sim 51$ MeV.

This value is very low.

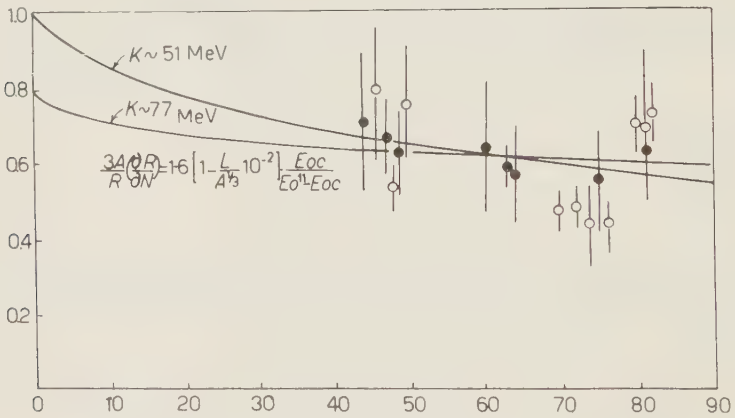

Fig. 4. – The experimental data represent the ratio between experimental and theoretical isotope shift constant, supposing the nuclei incompressible. The nuclei from left to right are Ru, Pd, Ag, Cd, In, Sn, Sm, Eu, Gd, Yb, Hf, W, Re, Os, Hg, Tl, Pb. The data for Sn, Sm, Eu, Gd (full circles) were corrected with respect to deformability. Those for In and Tl (also full circles) are obtained from h.f.s. anomalies. The two theoretical curves plotted versus charge number Z are computed supposing either no difference between proton and neutron distribution radii ($K_v \sim 51$ MeV) or a constant ratio ($K_v \sim 77$ MeV).

([15]) P. BRIX and H. KOPFERMANN: *Rev. Mod. Phys.*, **30**, 517 (1958).
([16]) N. J. IONESCO-PALLAS: *Compt. Rend. Acad. Sci. Paris*, 12 Jan. 1959.

Therefore, we were obliged to admit the Johnson-Teller effect and the inequality $R_Z < R_A$. The theoretical data must be corrected by the factor

$$\Xi \sim \left(\frac{1.089}{1.226}\right)^{2\varrho} \tag{6.1}$$

This correction increases the value of compressibility per nucleon K_v to ~ 77 MeV (see Fig. 4). This last value seems to be plausible, being consistent with the theoretical estimations of FEENBERG ([5]), BRUCKNER ([6]) and WILETS ([7]). A set of evaluations gives, on the other hand, much higher values ([17,18]), the theoretical results being very sensitive to thickness variations of nuclear surface.

We believe that the value estimated by us approximates well the order of magnitude of the K_v coefficient, because the ignoring of surface effects in formula (3.9) changes the results in an opposite sense to that given by overestimation of inner electrons screening.

([17]) R. A. BERG and L. WILETS: *Phys. Rev.*, **101**, 201 (1956).
([18]) K. A. BRUECKNER and J. L. GAMEL: *Phys. Rev.*, **105**, 1679 (1957).

RIASSUNTO (*)

Si espone una teoria generale dello spostamento degli isotopi in atomi pesanti tenendo conto degli effetti nucleari specifici e di quelli dovuti agli strati elettronici. Solo l'effetto di « spostamento dispari » non viene considerato data la particolare scelta dei nuclei (pari-pari). Il problema viene risolto in maniera rigorosa con la supposizione che la distribuzione dei protoni nei nuclei sia quella di Fermi. Per il calcolo della compressibilità si usa il modello di Wilets con qualche modifica. La discussione dei risultati per un largo campo di numeri di massa — compresa la zona di grande deformazione nucleare — dimostra la necessità dell'effetto Johnson-Teller e porta ad un valore della compressibilità di circa 77 MeV.

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A Class of Simple Field Theories and von Neumann's Infinite Direct Product Spaces (*).

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Summary. — The fixed scalar Boson field is reanalyzed in an attempt to clarify certain issues raised by Van Hove and Miyatake regarding the orthogonality of certain Hilbert spaces. The discussion is generalized to include a more general class of models of which the scalar boson field is a special case.

1. – Introduction.

In quantum field theory one is generally faced with the consideration of an infinite number of commuting (or anti-commuting) operators. This leads to the problem of handling vector spaces which are constructed from the direct product of an infinite number of Hilbert spaces. A mathematical theory for handling such a problem has been developed by von Neuman (1). Several simple problems have been related to von Neuman's theory. In particular, much attention has been devoted to the scalar boson field in scalar interaction with an infinitely heavy point source by Van Hove (2), Miyatake (3) and more recently by Albertoni and Duimio (4). However, some of the conclusions

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(1) J. von Neumann: *Compositio Matematica*, **6**, 1 (1938).

(2) L. Van Hove: *Acad. Roy. de Belgique, Bulletin*, **37**, 1055 (1951); *Physica*, **18**, 145 (1952).

(3) O. Miyatake: *Journ. Inst. of Polytechnics, Math. Series*, **2**, 89 (1951); **3**, 145 (1952).

(4) S. Albertoni and F. Duimio: *Nuovo Cimento*, **6**, 1193 (1957).

reached by VAN HOVE and MIYATAKE can be confusing because their consideration of the questions of orthogonality and normalization is incomplete in a sense to be described. The situation is further complicated by the fact that the interaction Hamiltonian used by the above authors has at best only a symbolic significance. Moreover, since the numbers 0 and ∞ are easy to come by in field theory, one must exercise extreme care to distinguish whether or not they have a real or just a formal significance.

In the present note we propose to rediscuss the scalar boson field problem in an attempt to clarify the issues raised by VAN HOVE and MIYATAKE. We will also discuss a more general class of simple models of which the scalar boson field is a special case.

Summaries of the pertinent results needed from von Neuman's work can be found in references ([3]) and ([4]). As a convenience, however, we also present an outline of some of the essential ideas of von Neuman's theory which will be called upon.

2. – Some results from von Neumann's theory.

Suppose we consider an infinite set of Hilbert spaces ([5]). We construct a product space by forming infinite products of vectors from each of the spaces. We treat such products as formal products without regard to convergence. Denote these products by $\prod_\alpha f_\alpha$. We impose the reasonable requirement that $\left(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha\right) = \prod_\alpha (f_\alpha, g_\alpha)$. Obviously, we have to worry about the convergence of such infinite products. In particular, we have $\|\prod_\alpha f_\alpha\| = \prod_\alpha \|f_\alpha\|$. Clearly we must be careful to choose the f_α with multiplicative factors which will not destroy the convergence of $\prod_\alpha \|f_\alpha\|$; even though $\|f_\alpha\| < \infty$, $\prod_\alpha \|f_\alpha\|$ may not converge.

If $\prod_\alpha \|f_\alpha\|$ converges then $\{f_\alpha\}$ is called a C sequence. If in addition $\sum_\alpha |\|f_\alpha\| - 1|$ converges, then $\{f_\alpha\}$ is called a C_0 sequence. Two C_0 sequences are considered equivalent if $\sum_\alpha |(f_\alpha, g_\alpha) - 1|$ is convergent. VON NEUMAN has shown the above equivalency to be reflexive, symmetric and transitive. Thus, mutually disjoint equivalence classes may be defined. Suppose Γ and Γ' are any two such disjoint equivalence classes. Then for all $f_\alpha \in \Gamma$ and $g_\alpha \in \Gamma'$, $\{f_\alpha\} \nsim \{g_\alpha\}$ and it may be shown that $\left(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha\right) = 0$. Note that the ortho-

([5]) Von Neuman's theory includes both denumerable and non-denumerable infinities. Accordingly, he has generalized the notion of convergence of sums and products to take this into account.

gonality between I' and I'' does not require the orthogonality of f_α and g_α. However, in case $\{f_\alpha\} \approx \{g_\alpha\}$, $\left(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha\right) = 0$ if and only if $(f_\alpha, f_\alpha) = 0$ for some α.

Von Neuman introduces a space of functionals based on C sequences and shows that this space is linear, metric and topologically complete. This space may be partitioned into mutually disjoint equivalence classes which thereby offer a subdivision into mutually orthogonal subspaces. Such subspaces are called incomplete direct product spaces.

Suppose I' is an equivalence class and $\{f_\alpha^0\}$ is a C_0 sequence in I' such that $|f_\alpha^0| = 1$. Let $f_{\alpha,\beta(\alpha)}$ be a c.n.o.s. (complete normalized orthogonal set) in the Hilbert space $\mathscr{H}_\alpha$. $(\beta(\alpha) = 0, 1, 2, \ldots$ and $f_{\alpha,0} = f_\alpha^0)$. The set of functionals formed by infinite products of the form $\prod_\alpha f_{\alpha,\beta(\alpha)}$ will form a c.n.o.s. for the IDP; I' (incomplete direct product space associated with I') whenever only a finite number of the $\beta(\alpha)$ in each infinite product is different from zero. The class of sets $\beta(\alpha)$ in each infinite product where only a finite number of the $\beta(\alpha) \neq 0$ will be denoted by F.

Denote by $\prod_\alpha{}_F f_{\alpha,\beta(\alpha)}$, those infinite products where only a finite number of the $\beta(\alpha) \neq 0$. Let Φ be any functional in the IDP; I'. There is a one to one correspondence between Φ and $\left(\Phi, \prod_\alpha{}_F f_{\alpha,\beta(\alpha)}\right)$ such that $\sum_{\beta(\alpha)\in F} \left|\left(\Phi, \prod_\alpha{}_F f_{\alpha,\beta(\alpha)}\right)\right|^2$ converges and for any two functionals Φ and Ψ in IDP; I',

$$(\Phi, \Psi) = \sum_{\beta(\alpha)\in F} \overline{\left(\Phi, \prod_\alpha{}_F f_{\alpha,\beta(\alpha)}\right)} \left(\Psi, \prod_\alpha{}_F f_{\alpha,\beta(\alpha)}\right) .$$

Moreover, the IDP; I' is linear, metric and topologically complete.

3. – Scalar boson field.

3·1. – The essential features of the problem are retained if we consider only one nucleon and assume it to be and infinitely heavy point source. Following Van Hove [2], the Hamiltonian is written as

$$H = H_B + gH_I , \tag{1}$$

where

$$H_B = \frac{1}{2}\int \mathrm{d}^3x \left(\pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\right) , \tag{2}$$

and

$$H_I - \beta\varphi(\boldsymbol{x}_s) . \tag{3}$$

β is the Dirac β matrix and $\boldsymbol{x}_s$ is the position of the source. Upon taking Fourier transforms we get

$$\varphi(\boldsymbol{x}) = \frac{1}{\sqrt{2V}} \sum_k \frac{1}{\sqrt{\omega}} \left(a_k \exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}\right] + a_k^+ \exp\left[-i\boldsymbol{k}\cdot\boldsymbol{x}\right]\right), \tag{4}$$

where $\omega = \sqrt{\mu^2 + k^2}$,

$$\pi = \dot{\varphi} = -\frac{i}{\sqrt{2V}} \sum_k \sqrt{\omega}\left(a_k \exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}\right] - a_k^+ \exp\left[-i\boldsymbol{k}\cdot\boldsymbol{x}\right]\right), \tag{5}$$

$$[a_k,\ a_{k'}^+] = \delta_{kk'}, \tag{6}$$

$$H_B = \sum_k \omega(a_k^+ a_k + \tfrac{1}{2}) \tag{7}$$

and

$$H_I = \frac{\beta}{\sqrt{2V}} \sum_k \frac{1}{\sqrt{\omega}} \left(a \exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}_s\right] + a_k^+ \exp\left[-i\boldsymbol{k}\cdot\boldsymbol{x}_s\right]\right). \tag{8}$$

Let

$$\gamma_k = \frac{g\beta}{\sqrt{V\omega^3}} \exp\left[-i\boldsymbol{k}\cdot\boldsymbol{x}_s\right] = \tau_k + i\sigma_k,$$

where

$$\tau_k = \frac{g\beta}{\sqrt{V\omega^3}} \cos\,\boldsymbol{k}\cdot\boldsymbol{x}_s; \qquad \sigma_k = \frac{-g\beta}{\sqrt{V\omega^3}} \sin\,\boldsymbol{k}\cdot\boldsymbol{x}_s. \tag{9}$$

The Hamiltonian may then be written as

$$H = \sum_k \omega\,\alpha_k^+ \alpha_k + \frac{1}{2}\sum_k \left(\omega - \frac{g^2}{V\omega^2}\right), \tag{10}$$

where

$$\alpha_k = a_k + \frac{1}{\sqrt{2}}\,\gamma_k. \tag{11}$$

We may go over to the harmonic oscillator form by introducing the Hermitean operators

$$q_k = \frac{1}{\sqrt{2}}\,(a_k + a_k^+); \qquad p_k = \frac{-i}{\sqrt{2}}\,(a_k - a_k^+), \tag{12}$$

which satisfy the canonical commutation rules

$$[p_k,\ q_{k'}] = -i\delta_{kk'}. \tag{13}$$

This yields

$$H = \frac{1}{2}\sum_k \omega[(p_k+\sigma_k)^2+(q_k+\tau_k)^2] - \frac{1}{2}\frac{g^2}{V}\sum_k \frac{1}{\omega^2}\,. \tag{14}$$

For each k we introduce a Schrödinger representation where $p_k = -i(\partial/\partial q_k)$ so that

$$\alpha_k = \frac{1}{\sqrt{2}}\left[(q_k+\tau_k)+\left(\frac{\partial}{\partial q_k}+i\sigma_k\right)\right].$$

The normalized eigenfunctions for each k will then be normalized Hermite functions. We have

$$\varphi^k_{n_k}(q_k) = \frac{1}{\sqrt{n!}}\,(\alpha_k^+)^{n_k}\varphi^k_0(q_k)\,, \tag{15}$$

where

$$\varphi^k_0(q_k) = \frac{1}{(\pi)^{\frac{1}{4}}}\exp\left[-i\sigma_k a q_k\right]\exp\left[-\tfrac{1}{2}(q_k+\tau_k)^2\right], \tag{16}$$

and

$$\alpha_k\varphi^k_0(q_k) = 0\,. \tag{17}$$

The orthonormal eigenfunctions of H will then be given by

$$\Phi\left(\{n_k\}\right) = \prod_k \varphi^k_{n_k}(q_k)\,. \tag{18}$$

The eigenfunctions of H_B are also given by (18) after setting $\tau_k = \sigma_k = 0$. Denote these eigenfunctions by $\Phi_B(\{n_k\})$. Suppose we consider those states Φ and Φ_B where $\sum_k n_k < \infty$ and form the inner product (Φ, Φ_B). Since $\sum_k n_k < \infty$ for both Φ and Φ_B, it is obvious that

$$\left(\Phi(\{n'_k\}),\ \Phi_B(\{n_k\})\right) = \text{const}\,\left(\Phi(0),\ \Phi_B(0)\right), \tag{19}$$

where $\Phi(0)$ and $\Phi_B(0)$ are the vacuum states for H and H_B respectively. Substituting (16) into $\Phi(0)$ and $(1/\pi^{\frac{1}{4}})\exp[-\frac{1}{2}q_k^2]$ into $\Phi_B(0)$, we get

$$\begin{aligned}\left(\Phi(\{n'_k\}),\ \Phi_B(\{n_k\})\right) &= \text{const}\prod_k \exp\left[-\frac{1}{4}(\sigma_k^2+\tau_k^2)\right]\exp\left[\frac{i\sigma_k\tau_k}{2}\right] = \\ &= \text{const}\prod_k \exp\left[\frac{-g^2}{4}\,\frac{1}{V\omega^3}\right]\exp\left[\frac{i\sigma_k\tau_k}{2}\right] = \\ &= \text{const}\exp\left[-\sum_k\left(\frac{g^2}{4}\,\frac{1}{V\omega^3}\right)-\frac{i\sigma_k\tau_k}{2}\right].\end{aligned} \tag{20}$$

Since $(g^2/4V)\sum_k 1/\omega^3$ is divergent, VAN HOVE and MIYATAKE conclude that $(\Phi, \Phi_B) = 0$ and hence the eigenstates of H are all orthogonal to the eigenstates of H_B. As we shall now show, consistency indicates that in calculating (Φ, Φ_B), $\prod_k \exp[-(g^2/4)(1/V\omega^3)]$ should be treated as a formal product rather than a numerical product; if it is treated as a numerical product, then consistency indicates that one should set $\Phi = \Phi_B = 0$ because in the representations used, each Φ and Φ_B contains as a factor, the infinite product of normalization factors, each one of which is less than unity. For example, both $\Phi(0)$ and $\Phi_B(0)$ contain $\prod_k 1/(\pi)^{\frac{1}{4}}$ as normalization factors; in a similar fashion $\prod_k \exp[-(g^2/4)(1/V\omega^3)]$ plays the role of a normalization factor of a vector in a particular representation of a complete, non-separable Hilbert space.

Let $\mathscr{H}$ denote the above mentioned non-separable Hilbert space. We define $\mathscr{H}$ as follows. Let I denote an infinite index set. With each $k \in I$ we associate the same Hilbert space h so that the product space will be based on infinitely many copies of h. Let h be the separable Hilbert space spanned by the orthonormal eigenfunctions of a number operator $A_k^+ A_k$. Denote these eigenfunctions by φ_{n_k}. There is a one to one correspondence between the products $\prod_{k\in I} \varphi_{n_k}$ and the sequences $\{n_k\}$, n_k a non-negative integer and $k \in I$. Let $\mathscr{H}'$ denote the space determined by all linear combinations of $\prod_{k\in I} \varphi_{n_k}$. Then $\mathscr{H}$ will correspond to the complete direct product space [1] determined by all convergent sequences of elements in $\mathscr{H}'$ [6]. The non-separability of $\mathscr{H}$ is due to the fact that the entire set of sequences formed from the elements in $\mathscr{H}'$ involve the entire non-denumerable set of $\{n_k\}$.

The $\prod_{k\in I} \varphi_{n_k}$ are examples of von Neuman's C_0 sequences. It is obvious that those $\prod_{k\in I} \varphi_{n_k}$ where $\{n_k\}$ differs from $\{n_k\}'$ only in a finite number of the n_k, the corresponding product vectors are inequivalent (and orthogonal) in von Neuman's sense. Therefore, the inequivalent, orthogonal, incomplete direct product spaces of $\mathscr{H}$ will be those corresponding to those classes of sequences $\{n_k\}$ where only a finite number of the n_k are different from each other. However, it should be noted that the operators $\sum_k A_k^+ A_k$ and $\sum_k \omega A_k^+ A_k$ are defined only in the subspace corresponding to $\sum_k n_k < \infty$, whereas the operators A_k and A_k^+ are defined everywhere as long as $n_k < \infty$.

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[6] It should be noted that in defining $\mathscr{H}$, we shall admit « ideal » functionals in the sense of Friedrichs. (See p. 155 of K. FRIEDRICHS: *Mathematical Aspects of the Quantum Theory of Fields*, New York 1953). Such functionals depend on the complete non-denumerable set of sequences $\{n_k\}$.

Now the sets of functions $\Phi_B(\{n_k\})$ and $\Phi(\{n_k\})$ each determine a representation of $\mathscr{H}$. Let $\mathscr{H}_B$ denote the representation determined by the former and $\mathscr{H}_g$ the representation determined by the latter. It is well known that $\mathscr{H}_g$ and $\mathscr{H}_B$ are connected by a formal unitary transformation (7). In the momentum representation this transformation is given by

$$(21)\qquad \begin{cases} T(\gamma_k) = \exp\left[\dfrac{1}{\sqrt{2}}\sum\limits_k (\gamma_k^+ a_k - \gamma_n a_k^+)\right], \\ \qquad = \exp\left[-\dfrac{1}{4}\sum\limits_k |\gamma_k|^2\right] \exp\left[-\dfrac{1}{\sqrt{2}}\sum\limits_k \gamma_k a_k^+\right] \exp\left[\dfrac{1}{\sqrt{2}}\sum\limits_k \gamma_k^+ a_k\right]. \end{cases}$$

Using the first form of T, easily verifies that $\alpha_k = T a_k T^+$, $H = T H_B T^+$ and $\sum\limits_k \alpha_k^+ \alpha_k = T\left(\sum\limits_k a_k^+ a_k\right) T^+$. Using the second form of T one verifies by formal power series expansion that $\Phi(\{n_k\}) = T\Phi_B(\{n_k\})$ (8). Note that $\prod\limits_k \exp[-\frac{1}{4}|\gamma_k|^2]$ will be a coefficient of each term in the power series expansion. Now the above expansion is an expansion in the Hilbert vector space corresponding to $\mathscr{H}_B$. The norm is then determined by the infinite series of the squared magnitudes of the expansion coefficients. The product $\prod\limits_k \exp[-g^2/4V\omega^3]$ must then be retained and treated formally in the expansion so that when the norm is formed, the series will converge to unity. This is entirely analogous to treating $\prod\limits_k 1/(\pi)^{\frac{1}{2}}$ as a formal product when one normalizes in the Hilbert functional space.

As an example consider

$$(22)\qquad \prod_k \frac{1}{(\pi)^{\frac{1}{2}}} \exp[-i\sigma_k q_k] \exp\left[-\frac{1}{2}(q_k+\tau_k)^2\right] = \\ = \exp\left[-\frac{1}{4}\sum_k |\gamma_k|^2\right] \sum_n \frac{(-1)^n}{n!}\left(\sum_k \frac{\gamma_k}{\sqrt{2}} a_k^+\right)^n \prod_k \varphi_B^k(q_k),$$

with

$$\left(\Phi_B(\{n_k\}), \Phi(0)\right) = \text{const} \prod_k \exp[-\tfrac{1}{4}|\gamma_k|^2].$$

It is easily verified that $\int |\Phi|^2 \mathrm{d}q - \sum |(\Phi_B, \Phi(0))|^2 - 1$. Therefore, if one

(7) For a thorough study of this transformation see Sect. **14** of K. Friedrichs, loc. cit. The term formal is used above to emphasize the fact that the unitarity depends on treating $\prod\limits_k \exp[-g^2/4V\omega^3]$ as a formal product.

(8) In using $T(\gamma_k)$, we will not worry about taking care of the extra infinite constant which appears in the interaction Hamiltonian. If desired, the original interaction Hamiltonian (eq. 8) could be redefined with the infinite c number term in it.

treats $\prod_k \exp[-\frac{1}{4}|\gamma_k|^2]$ as a numerical product rather than as a formal normalizing factor, consistency indicates that $\prod_k 1/(\tau)^{\frac{1}{2}}$ also be treated as a numerical product. But this amounts to saying that $\Phi = \Phi_B = 0$.

The above situation always arises when one forms a direct product space from infinitely many copies of a single Hilbert space because each vector carries with it a normalizing factor determined by the norm in the Hilbert space. However, if one only requires that Φ and Φ_B be normalizable with respect to some measure, then one should treat the infinite product of normalizing factors as a formal product until norms or probability distributions are calculated.

The situation which we have just described is rather remarkable for the following reason. We have seen how $T(\gamma_k)$ can be given a meaning in the complete non-separable space $\mathscr{H}$ such that the states $\Phi(\{n'_k\})$ are related to the states $\Phi_B(\{n_k\})$ by a unitary transformation. Nevertheless, the two states are orthogonal according to von Neuman's theory. To show this one may directly apply von Neuman's convergence criteria; note that von Neuman's criteria cover both the case when k is a discrete as well as a continuous variable. One first shows the divergence of $\sum_k |\exp[-\frac{1}{4}|\gamma_k|^2] \exp[i(\sigma_k\tau_k/2)] - 1|$. This establishes that $\Phi(\{n'_k\})$ and $\Phi_B(\{n_k\})$ are associated with inequivalent C_0 sequences. The conditions of von Neuman's orthogonality theorem will then be satisfied and one conclude that $\Phi(\{n'_k\})$ and $\Phi_B(\{n_k\})$ are orthogonal.

It is important to note that if k is a continuous variable, then (apart from the usual problems with commutation and other relations) the above orthogonality holds no matter what one has for γ_k. This is because $\sum_k f_k$ converges if and only if $f_k \neq 0$ only for a finite or enumerably infinite number of k. The preceding remark illustrates the significance of formulating the present and similar discussions with k as a discrete variable.

As Wightman and Schweber [9] have pointed out, the inequivalence of the domains of H and H_B is related to the inequivalence of the representation of the commutation relations by the operators $\alpha_k = a_k + (k/\sqrt{2})$ and a_k. The representation by the operators α_k corresponds to what they have called a continuous representation. We may gain further insight into the underlying causes of the above inequivalence from the following approach.

When the transformation $T(\gamma_k)$ is applied, we have the choice of considering the co-ordinate system to be transformed or else considering the transformation to produce new operators in the original co-ordinate system. From the former point of view we would consider the transformation $\mathscr{H}_g = T\mathscr{H}_B$ as producing a change of basis vectors. Then α_k, H, and $\sum \alpha_k^+ \alpha_k$ would be

[9] A. S. Wightman and S. S. Schweber: *Phys. Rev.*, **98**, 812 (1955).

the images of a_k, H_B and $\sum a_k^+ a_k$ respectively in the new co-ordinate system. Moreover, the transformation leaves the matrix elements of a_k, H_B and $\sum a_k^+ a_k$ unchanged. However, the situation is not as simple from the latter point of view. As indicated above, H_B and $\sum a_k^+ a_k$ are defined only in the subspace of $\mathscr{H}_B$ corresponding to all $\{n_k\}$ such that $\sum n_k < \infty$ whereas $T(\gamma_k)$ is defined over all $\mathscr{H}_B$. Therefore, the possibility exists that the transformation produces operators which are no longer defined in the subspace of $\mathscr{H}_B$ corresponding to $\sum n_k < \infty$. In the present model, the above possibility is realized for H and $\sum \alpha_k^+ \alpha_k$ because of the particular functional dependence of γ_k and ω.

The application of $T(\gamma_k)$ to H_B produces $H_B + gH_I$ and H_I is an undefineable operator (even in $\mathscr{H}$) ([2,3,10]). This is tied up with the fact that the exponent in $T(\gamma_k)$ is undefineable and the only way in which we could give a meaning to $T(\gamma_k)$ in $\mathscr{H}$ is to allow « ideal elements » in $\mathscr{H}$ ([6]). In fact, from the point of view of keeping the co-ordinate system fixed, all elements in $\mathscr{H}_g = T\mathscr{H}_B$ are « ideal » elements.

The significance of these ideal elements may be better understood by considering certain probabilities. Particles associated with H will be called physical particles and those associated with H_B will be called bare particles. Physical particle numbers will be primed and bare particle numbers unprimed. Consider the physical vacuum. Let $P(n, 0)$ denote the probability of finding n bare particles after the system is in the physical vacuum state. From (22) we obtain the Poisson distribution

$$P(n, 0) = (1/n!)\, \gamma^n \exp[-\gamma], \tag{23}$$

where $\gamma = \frac{1}{2} \sum_k |\gamma_k|^2$. In a similar fashion, the form of (22) shows that if $P(0, n')$ is the probability of finding n' physical particles after the system is in the bare vacuum state, then

$$P(0, n') = P(n, 0)\,. \tag{24}$$

Upon calculating the expectation value $\left(\Phi(0), \sum a_k^+ a_k \Phi(0)\right) = \sum_n n P(n, 0)$, we get

$$\left(\Phi(0), \sum a_k^+ a_k \Phi(0)\right) = \gamma\,. \tag{25}$$

([10]) By undefineable, we mean $\|H_I \Phi\| = \infty$ for all $\Phi \in \mathscr{H}$. The above statement would be true even if one introduced nucleon creation and annihilation operators and wrote H_I in the form $\int d^3x \psi^+ \psi \varphi$. The main difference in the latter case is that the nucleon number operator appears in $T(\gamma_k)$ and the vacuum state is common to both H_B and H.

Since γ is a divergent sum, one may say that $P(n, 0) = P(0, n') = 0$ for $n, n' < \infty$ and $P(n, 0) = P(0, n') = 1$ for $n, n' = \infty$. This last result is an expression of the respective myriotic character ([11]) of the physical particle field with respect to $\mathscr{H}_B$ and the bare particle field with respect to $\mathscr{H}_g$.
It is also indicative of the continuous nature of the representation of the commutation relations by the operators $a_k \pm \gamma_k$ ([9,12]).

3·2. – We will conclude our discussion of the scalar boson field with the discussion of certain transition probabilities. In the Heisenberg picture we have $\dot{a}_k = i[H, a_k]$ so that

$$a_k(t) = \left(a_k + \frac{\gamma_k}{\sqrt{2}}\right) \exp\left[-i\omega t\right] - \frac{1}{\sqrt{2}}\gamma_k = \alpha_k \exp\left[-i\omega t\right] - \frac{1}{\sqrt{2}}\gamma_k . \tag{26}$$

Following FRIEDRICHS ([13]) we write

$$a_k(t) = \exp\left[-i\omega t\right]\left[a_k + \frac{1}{\sqrt{2}}\gamma_k - \frac{1}{\sqrt{2}}\gamma_k(t)\right], \tag{27}$$

where $\gamma_k(t) = \gamma_k \exp\left[i\omega t\right]$. Friedrichs' adjusted operators are then defined from

$$A_k(t) = \exp\left[i\omega t\right] a_k(t) = a_k + \frac{1}{\sqrt{2}}\left(\gamma_k - \gamma_k(t)\right) . \tag{28}$$

We work with the adjusted operators because we are interested in the transition (with the presence of the interaction Hamiltonian) from an eigenstate of $H_B(t_0)$ to an eigenstate of $H_B(t)$; this involves the transformation from $a_k(t_0)$ to $A_k(t)$ rather than from $a_k \exp\left[-i\omega(t - t_0)\right]$ to $a_k(t)$.

In view of the discussion in Section 3·1 to Eq. (21), we see from the form of Eq. (28) that $A_k(t) = T a_k T^{+}$, $H_B(t) = T H_B T^{+}$, and $\sum_k A_k^{+} A_k = T \sum_k a_k^{+} a_k T^{+}$ where $T = T(\gamma_k - \gamma_k(t))$. Thus

$$N(t) = \sum_k A_k^{+} A_k = N + \frac{1}{\sqrt{2}} \sum_k \left[\left(\gamma_k^{+} - \gamma_k^{+}(t)\right) a_k + \left(\gamma_k - \gamma_k(t)\right) a_k^{+}\right] + \\ + \frac{1}{2} \sum_k \left|\gamma_k - \gamma_k(t)\right|^2 , \tag{29}$$

$$H_B(\tau) = H_B + \frac{1}{\sqrt{2}} \sum_k \omega \left[\left(\gamma_k^{+} - \gamma_k^{+}(t)\right) a_k + \left(\gamma_k - \gamma_k(t)\right) a_k^{+}\right] + \frac{1}{2} \sum_k \omega \left|\gamma_k - \gamma_k(t)\right|^2 . \tag{30}$$

([11]) K. FRIEDRICHS: loc. cit., Sect. **14**.

([12]) A similar discussion may be found in a paper by R. HAAG: *Dan. Mat. Fys. Medd.*, **29**, 12 (1954).

([13]) K. FRIEDRICHS: loc. cit., Sect. **15**.

$H_I(t)$ in the Heisenberg picture is obtained by replacing a_k in H_I by $a_k(t)$ so that

$$(31) \quad H_I(t) = \frac{1}{\sqrt{2}} \sum_k \omega [\gamma_k^+(t) a_k + \gamma_k(t) a_k^+] + \frac{1}{2} \sum_k \omega \gamma_k^+ \gamma_k (\exp[i\omega t] + \exp[-i\omega t] - 2).$$

Moreover, if Φ_B is an eigenstate of H_B, then

$$(32) \quad \Phi_B(t) = T(\gamma_k - \gamma_k(t)_k \Phi_B$$

is an eigenstate of $H_B(t)$. In the next section we will show that $\Phi_B(t)$ is a state in a picture where the time dependence of both state vectors and operators is determined by $H_I(t)$.

If Φ_B is the vacuum state at t_0, then it follows from Eq. (32) that the probability that $n(t)$ particles have been created in the time $t - t_0$ is given by the Poisson distribution,

$$(33) \quad P(n(t), 0) = (1/n!) \gamma_t^n \exp[-\gamma_t],$$

where $\gamma_t = \frac{1}{2} \sum_k |\gamma_k - \gamma_k(t - t_0)|^2$. Simple calculation shows that

$$(34) \quad \gamma_t = \sum_k |\gamma_k|^2 (1 - \cos \omega(t - t_0)) = 2 \sum_k |\gamma_k|^2 \sin^2 \frac{\omega(t - t_0)}{2}.$$

Now $\sum_k |\gamma_k|^2 \cos \omega(t - t_0)$ is convergent so γ_t diverges. Consequently, as in Section 3·1, $P(n(t), 0) = 0$ for finite n and $\langle N(t) \rangle_{\text{vac}} = \gamma_t = \infty$. Neglecting the divergence of γ_t, however, it should be noted that since $\operatorname{Lim}_{t - t_0 \to \infty} \sum_k |\gamma_k|^2 \cdot \cos \omega(t - t_0) = 0$, then $\operatorname{Lim}_{t - t_0 \to \infty} \langle N(t) \rangle_{\text{vac}} = \sum_k |\gamma_k|^2 = 2\gamma$, where γ is as defined below eq. (23). It is also interesting to note that even though $\operatorname{Lim}_{t - t_0 \to \infty} \Phi_B(t)$. in eq. (32) may not exist, the asymptotic transition probabilities may exist This would be the case in the present situation if $\sum_k |\gamma_k|^2 < \infty$.

The fact that $\operatorname{Lim}_{t - t_0 \to \infty} \langle N(t) \rangle_{\text{vac}} = 2\gamma$ rather than γ as in Section 3·1 will be discussed in the next section. For the present, we simply note that $T(\gamma_k - \gamma_k(t))$ as well as $T(\gamma_k)$ produces an « ideal » element in $\mathscr{H}_B$ when operating on a proper element of $\mathscr{H}_B$. Furthermore, $N(t)$, $H_B(t)$ and $H_I(t)$ are indefineable in $\mathscr{H}_B$.

4. – A general class of simple models.

4·1. – The scalar Boson field model which we have just discussed is a special case of a more general class which may be constructed in the following way. For specificness, consider field operators in momentum space. Take any

number of Fermian or Boson fields and construct a free field Hamiltonian by forming the sum of the individual free field Hamiltonians. Form multilinear products of any number of the above field operators multiplied by arbitrary energy dependent factors. Take a linear combination of such products. Form the difference between the linear combination and its Hermitian conjugate. This difference is anti-Hermitian. Sum over all momentum variables concerned. The sum is also anti-Hermitian so the exponential of the sum will be unitary ([14]). Denote the sum by S. Then the operator e^S will transform the free field Hamiltonian, H_0, into

$$H = e^S H_0 e^{-S} = H_0 + \sum_{n=1}^{\infty} \frac{1}{n!} [S, H_0]_n , \tag{35}$$

where $[S, H_0]_n$ is the usual n-th order commutator. We now define the interaction Hamiltonian to be

$$H_I = \sum_{n=1}^{\infty} \frac{1}{n!} [S, H_0]_n . \tag{36}$$

For the scalar Boson field, S is easily identified from eq. (21). In that model the infinite series of commutators cuts-off after $n = 2$. As another simple illustrative example, consider the following S constructed from Lee model ([15]) operators. Let

$$S(\gamma_k) = g \sum_{P_V P_N k} (\gamma_k^+ \psi_{P_V}^+ \psi_{P_N} a_k - \gamma_k \psi_{P_N}^+ a_k^+ \psi_{P_V}) \, \delta(P_N + k - P_V) , \tag{37}$$

where P , P and k denote V-particle, N-particle and Boson momentum respectively, g is a coupling constant and γ_k is a c-number depending on some inverse power of ω (the Boson energy) and the normalization volume V. As in the Lee model, the V and N particles will be Fermions, the Bosons will be called θ-particles. The operator

$$T(\gamma_k) = \exp [S(\gamma_k)] \tag{38}$$

will then be the unitary operator constructed according to the prescription given above. If

$$H_0 = m \sum_{P_V} \psi_{P_V}^+ \psi_{P_V} + m \sum_{P_N} \psi_{P_N}^+ \psi_{P_N} + \sum_k \omega a_k^+ a_k \tag{39}$$

([14]) Note that instead of using the difference between the linear combination and its Hermitian conjugate, we could have used the sum of the two multiplied by $\sqrt{-1}$ in order to form a unitary operator; but our purpose will be served by the prescription given above.

([15]) T. D. Lee: *Phys. Rev.*, **95**, 1329 (1954).

where m is the mass of both the V and N particles, then

$$H = e^{S} H_0 e^{-S} = m \sum_{P_V} \psi^{+}_{P_V} \psi_{P_V} + m \sum_{P_N} \psi^{+}_{P_N} \psi_{P_N} + e^{S} \sum_{k} \omega a^{+}_{k} a_k e^{-S} , \tag{40}$$

since $[a^{+}_{k} a_{k'}, [s(\gamma_k)]] = \psi^{+}_{P_N} \psi_{P_N}, s(\gamma_k)] = -[\psi^{+}_{P_V} \psi_{P_V}, s(\gamma_k)]$, where $s(\gamma_k)$ is the summand in $S(\gamma_k)$. Furthermore, the operators $N_V + N_0 = \sum_{P_V} \psi^{+}_{P_V} \psi_{P_V} + \sum_{k} a^{+}_{k} a_k$ and $N_V - N_N = \sum_{P_V} \psi^{+}_{P_V} \psi_{P_V} - \sum_{P_N} \psi^{+}_{P_N} \psi_{P_N}$ commute with H (because of the above mentioned commutators with $s(\gamma_k)$) so that we have the same conservation laws encountered in the Lee model.

The interaction Hamiltonian is

$$H_I = h_I + \sum_{n=2}^{\infty} \frac{1}{n!} [S, H_0]_n , \tag{41}$$

where

$$h_I = [S, H_0] = -g \sum_{P_V P_N k} \omega (\gamma_k \psi^{+}_{P_N} a^{+}_{k} \psi_{P_V} + \gamma^{+}_{k} \psi_{P_V} \psi_{P_N} a_k) \delta(P_N + k - P_V) , \tag{42}$$

will be recognized as essentially the Lee model interaction Hamiltonian.

The vacuum state, the one N-particle state and the one θ-particle state are invariant under the application of $T(\gamma_k)$ so that they will be simultaneous eigenstates of both H and H_0. However, all other states are transformed. For example, for the one V-particle state we have

$$\exp[S] |P_V\rangle = |P_V\rangle + \sum_{n=1}^{\infty} \frac{1}{n!} [S(\gamma_k)]^n |P_V\rangle = \tag{43}$$

$$= \cos g\lambda |P_V\rangle - \frac{\sin g\lambda}{\lambda} \Big(\sum_{k, P_N} \gamma_k \delta(P_N + k - P_V) |P_N, k\rangle \Big) ,$$

where $\lambda = \Big(\sum_{k} |\gamma_k|^2 \Big)^{\frac{1}{2}}$. For the state with one N and one θ particle we have

$$\exp[S] |k, P_n\rangle = |k, P_N\rangle + \frac{(\cos g\lambda - 1)}{\lambda^2} \sum_{P_N k'} \gamma^{+}_{k} \gamma_{k'} \cdot \tag{44}$$

$$\cdot \delta(P_N + k - P'_N - k') |k', P'_N\rangle + \frac{\sin g\lambda}{\lambda} \sum_{P_V} \gamma^{+}_{k} \delta(P_N + k - P_V) |P_V\rangle .$$

It is clear from eqs. (43) and (44) that just as with the scalar Boson field (and with the Lee model as well), convergence problems are associated with $\sum_{k} |\gamma_k|^2$. Because of the above mentioned particle number conservation laws, the transformation $T(\gamma_k)$ always leaves us in the subspaces of $\mathscr{H}_0$ corresponding

to a finite number of particles; but if $\lambda = \infty$, it is seen from the $\cos g\lambda$ term in eq. (43) that not all of the matrix elements of H and e^S will be well defined in that subspace; although vectors transformed by e^S will still be normalizable in a formal sense.

4·2. – The discussion which follows may to some extent be considered a generalization of Friedrichs' discussion of transition probabilities ([16]).

Consider a Heisenberg picture with the total Hamiltonian given by eq. (35). Then H is a constant of the motion and in general we may write

$$H = \exp[S(t)] H_0(t) \exp[-S(t)] = H_0(t) + \sum_{n=1}^{\infty} \frac{1}{n!} [S(t), H_0(t)]_n , \tag{45}$$

where $S(t)$ and $H_0(t)$ are the same functions of the field operators at $t > 0$ or $t < 0$ as at $t = 0$; however, the field operators must correspond to time t and be solutions to the Heisenberg equations of motion. Let $a_k(t)$ (where k denotes momentum) be any one of the field operators which determines $H_0(t)$. Then eq. (45) implies that $a_k(t)$ is transformed by $\exp[S(t)]$ into

$$\alpha_k(t) = \exp[S(t)] a_k(t) \exp[-S(t)] . \tag{46}$$

But if H is the total Hamiltonian then,

$$\alpha_k(t) = \alpha_k \exp[-i\omega t] = \exp[S] a_k \exp[-S] \exp[-i\omega t] , \tag{47}$$

where α_k, $\exp[S]$ and a_k correspond to $t = 0$ and ω is the energy associated with the particular field evaluated for momentum k.

From eqs. (46) and (47) we deduce that

$$\alpha_k(t) = \exp[-S(t)] \exp[S] a_k \exp[-S] \exp[S(t)] \exp[-i\omega t] . \tag{48}$$

It then follows equally from eq. (48) or eq. (45) that

$$H_0(t) = \exp[-S(t)] \exp[S] H_0 \exp[-S] \exp[S(t)] , \tag{49}$$

and

$$\left\{ \begin{aligned} H_I(t) &= \exp[-S(t)] \sum_{n=1}^{\infty} \frac{1}{n!} [S(t), \exp[s] H_0 \exp[-S]]_n \exp[S(t)] = \\ &= \exp[-S(t)] \sum_{n=1}^{\infty} \frac{1}{n!} [S(t), H]_n \exp[S(t)] = \\ &= i \exp[-S(t)] \sum_{n=0}^{\infty} \frac{1}{(n+1)!} [S(t), \dot{S}(t)]_n \exp[S(t)] . \end{aligned} \right. \tag{50}$$

([16]) K. O. Friedrichs: loc. cit., Sect. **15**.

We will now show that state vectors and operators are transformed by $U(t) = \exp[-S(t)] \exp[S]$ to a picture where the time dependence of both state vectors and operators is governed by $H_I(t)$, where $H_I(t)$ is the Heisenberg picture interaction Hamiltonian. State vectors in $\mathscr{H}_0$, at $t = 0$ are transformed into

$$\Phi_t = \exp[-S(t)] \exp[S] \Phi_0 . \tag{51}$$

From the last line of eq. (50) we then deduce that

$$\frac{\partial \Phi_t}{\partial t} = i H_I(t) \Phi_t . \tag{52}$$

Operators defined in $\mathscr{H}_0$ at $t = 0$ are transformed into

$$\Phi_t = \exp[-S(t)] \exp[S] \Phi_0 \exp[-S] \exp[S(t)] . \tag{53}$$

$$\frac{\partial \Phi_t}{\partial t} = -\exp[-S(t)] \sum_{n=0}^{\infty} \frac{1}{(n+1)!} [S(t), \dot{S}(t)]_n \exp[S] \Phi_0 \exp[-S] \exp[S(t)] + \Phi_t \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} [S(t), \dot{S}(t)]_n .$$

We again substitute from eq. (50) so that the first term on the right above becomes $i H_I(t) \Phi_t$. For the second term on the right we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} [S(t), \dot{S}(t)]_n = i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [S(t), H]_n .$$

Now

$$\exp[-S(t)] H \exp[S(t)] = H + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [S(t), H]_n = H_0(t) ,$$

so

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} [S(t), \dot{S}(t)]_n = -i H_I(t) , \tag{54}$$

and

$$\frac{\partial \Phi_t}{\partial t} = i [H_I(t), \Phi_t] . \tag{55}$$

Note that the Schrödinger equation, eq. (52), differs from that in the usual interaction picture in that $i\dot{\Phi}_t = -H_I \Phi_t$ rather than $i\dot{\Phi}_t = H_I \Phi_t$.

Upon comparing eq. (51) with the iteration solution to eq. (52), we see that

$$U(t) = \exp[-S(t)\exp[S] = P\exp\left[i\int_0^t H_I(t')\,\mathrm{d}t'\right], \tag{56}$$

where P denotes the usual P ordering operator. We see from eq. (56) that if the iteration solution to eq. (52) exists, it will correspond to a unitary transformation from the state at $t=0$ to time t.

There is a simple physical interpretation for $U(t, t_0)$. Let Φ_0 be an eigenstate of H_0 at $t=t_0$. We see from eq. (51) that the eigenstates of $H_0(t)$ are connected to Φ_0 only by first transforming to $\Psi_0 = \exp[S]\Phi_0$, where Ψ_0 is an eigenstate of H. Therefore, we may think of $U(t, t_0)$ as turning on the interaction on and *leaving it on*, thereby creating bare particles some of which clothe the physical particles. In the above sense, $\exp[S]$ may be thought of as generating a clothing transformation ([17]). The above result may also be expressed by noting that the bare particle number operator at time t is

$$N_0(t) = \exp[-S(t)\exp[S]N_0\exp[-S]\exp[S(t)]] = \tag{57}$$

$$= \exp[-S(t)]N\exp[S(t)] = N + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}[S(t), N]_n\,,$$

where N is the physical particle number operator. We see immediately from eq. (57) that the bare particle expectation values of $N_0(t)$ will generally be different from those of N, but the physical particles are generally always present; part of the bare particles created after time t will generally always be clothing for physical particles.

S and $S(t)$ are easily identified in the case of the scalar Boson field. The application of the preceding results to that case is immediate. In particular, we see from the above remarks how to interpret the result that $\lim_{t-t_0\to\infty}\langle N_0(t, t_0)\rangle_{\mathrm{vac}} = 2\gamma$ while $\langle N\rangle_{\mathrm{vac}} = \gamma$.

In general, since $S(t)$ is a Heisenberg operator, we have $S(t) = \exp[iHt]\cdot s\exp[-iHt]$, $\exp[S(t)] = \exp[iHt]\exp[S]\exp[-iHt]$ and

$$U(t) = \exp[iHt]\exp[-S]\exp[-iHt]\exp[S]\,. \tag{58}$$

We see from Eq. (58) that the transitions we have been considering do not refer to scattering in the usual sense because $U(0, -t) = U(0, t)$. The models under consideration describe essentially only creation and annihilation of par-

([17]) O. W. Greenberg and S. S. Schweber: *Nuovo Cimento*, **8**, 378 (1958).

ticles. However it may be noted that by cutting the infinite series of n-th order commutators in eq. (35) at finite n, one may sometimes obtain models which describe particle scattering. For example, if one retains only h_I in eq. (41), one is left with the Lee model Hamiltonian, which admits scattering state solutions.

* * *

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RIASSUNTO (*)

Il campo fisso scalare bosonico è stato rianalizzato per tentare di chiarire alcuni punti segnalati da Van Hove e Miyatake riguardanti l'ortogonalità di certi spazi Hilbertiani. La discussione viene generalizzata per includere una categoria più ampia di modelli, di cui il campo scalare bosonico è un caso particolare.

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Gravitational Radiation (*).

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Summary. — A method is established to solve the Einstein equations for a system of freely gravitating pole particles, by successive approximations. It is shown how one can choose the solution that represents purely outgoing waves. It is then found that the fifth order correction to the acceleration involves a non-conservative term: energy is lost, by gravitational radiation, in an amount exactly equal to that predicted by the linearized theory. This can also be shown by directly computing the loss of mass of the system. We then turn to examine the validity of the linearized theory: it is shown that it cannot correctly describe the field at very large distances from the sources, but nevertheless it gives the right result for the radiated energy.

1. – Introduction.

Since the early years of the General Relativity Theory, it has been known that the linearized field equations have wavelike solutions, and that the corresponding energy-momentum pseudotensor of the gravitational field has components representing an energy flux [1]. This result, however, was treated with caution, since the field equations are really non-linear. Later, approximation methods were discovered, which took account of the non-linearity, and gave the equation of motion of particles. The situation then became quite chaotic. Some authors [2-4] claimed that there was no radiation reaction,

(*) Partly supported by the U.S. Air Force, through ARDC.

(1) L. Landau and E. Lifshitz: *The Classical Theory of Fields* (Cambridge, Mass., 1951), p. 331.

(2) L. Infeld and A. E. Scheidegger: *Can. Journ. Math.*, **3**, 195 (1951).

(3) A. E. Scheidegger: *Phys. Rev.*, **82**, 883 (1951); *Rev. Mod. Phys.*, **25**, 451 (1953).

(4) L. Infeld: *Ann. Phys.*, **6**, 341 (1959).

others [5,6] found a gravitational damping—which agreed, but only qualitatively, with the linearized theory—and still others [7,8] found a gravitational « antidamping », *i.e.* an energy gain!

One of the main causes of this trouble doubtless was the complexity of the field equations: when extremely cumbersome expressions have to be handled, the physical meaning of the various terms becomes unclear. The first step toward the solution of this problem is therefore to develop an approximation technique which minimizes the computational labour, and thus also the risk of errors. The method that was recently developed by the author seems well fitted for this purpose. It needs, however, some modifications, as will be shown in Section **4**. For the sake of completeness, let us briefly recall it.

2. – Approximation procedure.

One takes as field variables the contravariant densities $\mathfrak{g}^{\mu\nu}$ and one expands them into a series (*)

$$\mathfrak{g}^{\mu\nu} = \underset{0}{\mathfrak{g}}^{\mu\nu} + \underset{1}{\mathfrak{g}}^{\mu\nu} + \underset{2}{\mathfrak{g}}^{\mu\nu} + \cdots ,$$

where each term is by one order of magnitude smaller than the previous one. One choose quasi-Galilean co-ordinates $(\underset{0}{\mathfrak{g}}^{\mu\nu} - \eta^{\mu\nu})$ subjected to the harmonic condition $\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0$. The solution behaving as purely outgoing waves is then unique [9].

The Einstein equations how read, in natural units:

$$\nabla^2 \mathfrak{g}^{\mu\nu} - \ddot{\mathfrak{g}}^{\mu\nu} = -16\pi \mathfrak{g} \mathfrak{T}^{\mu\nu} + \Theta^{\mu\nu} , \tag{1}$$

where $\mathfrak{g} = (-\text{Det. } \mathfrak{g}^{\alpha\beta})^{\frac{1}{2}}$ and $\Theta^{\mu\nu}$ is quadratic in $(\mathfrak{g}^{\mu\nu} - \eta^{\mu\nu})$. When (1) is expanded into the various orders (in order to compute $\underset{n}{\mathfrak{g}}^{\mu\nu}$), $\Theta^{\mu\nu}$ thus depends only on $\underset{m}{\mathfrak{g}}^{\mu\nu}$, $m < n$, and is a known quantity. The compatibility of (1) with the harmonic condition $\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0$ follows from the equations of motion [10], which are most conveniently written as $(\mathfrak{g}\mathfrak{T}^{\mu\nu})_{,\nu} = 0$, or

$$(\mathfrak{g}\mathfrak{T}^{\mu\nu})_{,\nu} = \mathfrak{g}\mathfrak{T}^{\alpha\beta}[\mathfrak{g}_{\alpha\gamma}\mathfrak{g}^{\mu\gamma}{}_{,\beta} + \tfrac{1}{2}\mathfrak{g}^{\mu\gamma}(\tfrac{1}{2}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\varrho\sigma} - \mathfrak{g}_{\alpha\varrho}\mathfrak{g}_{\beta\sigma})\mathfrak{g}^{\varrho\sigma}{}_{,\gamma}] , \tag{2}$$

where $\mathfrak{g}_{\mu\nu} = \mathfrak{g}^{-1}\mathfrak{g}_{\mu\nu}$.

(5) P. Havas: *Phys. Rev.*, **108**, 1351 (1957).
(6) A. Trautman: *Bull. Acad. Polon. Sc.*, **6**, 627 (1958).
(7) N. Hu: *Proc. Roy. Irish Acad.*, A **51**, 87 (1947).
(8) A. Peres: *Nuovo Cimento*, **11**, 644 (1959).
(*) Greek indices run from 0 to 3, Latin indices from 1 to 3.
(9) V. A. Fock: *Teoria Proctranstva, Vremeni i Tyagotenia* (Moscow, 1955), p. 441.
(10) F. Hennequin: *Thèse* (Paris, 1956).

For pole particles, localized at $\overset{A}{\xi}{}^k = \overset{A}{\xi}{}^k(t)$, one takes [11]

$$\mathfrak{g}\mathfrak{T}^{\mu\nu} = \sum_A \overset{A}{M}\overset{A}{v}{}^\mu \overset{A}{v}{}^\nu \overset{A}{\delta}(\overrightarrow{x-\xi}) , \tag{3}$$

where $v^\mu = \mathrm{d}\xi^\mu/\mathrm{d}t$ and M is a function of time. (We call M the « effective gravitational mass ». It can be shown [11] that the « intrinsic mass » $m_0 = M\mathfrak{g}^{-1}\mathrm{d}s/\mathrm{d}t$ is constant). The equations of motion then read

$$(\dot{M}v^\mu) = Mv^\alpha v^\beta [\mathfrak{g}_{\alpha\gamma}\mathfrak{g}^{\mu\gamma}{}_{,\beta} + \tfrac{1}{2}\mathfrak{g}^{\mu\gamma}(\tfrac{1}{2}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\varrho\sigma} - \mathfrak{g}_{\alpha\varrho}\mathfrak{g}_{\beta\sigma})\mathfrak{g}^{\varrho\sigma}{}_{,\gamma}] , \tag{4}$$

where the bracket has to be computed at the position of the particle under consideration, and all the singular terms in it have to be neglected.

All that remains now to do is to solve (1) at the various orders, so as to get the $\mathfrak{g}^{\mu\nu}$ which appear in the right hand side of (4). For this purpose, it is convenient to write

$$\mathfrak{g}^{\mu\nu} = \mathfrak{h}^{\mu\nu} + \mathfrak{s}^{\mu\nu} ,$$

where

$$\nabla^2\mathfrak{h}^{\mu\nu} - \ddot{\mathfrak{h}}^{\mu\nu} = -16\pi\mathfrak{g}\mathfrak{T}^{\mu\nu} \tag{5}$$

is readily solved by the use of Liénard-Wiechert potentials, and

$$\nabla^2\mathfrak{s}^{\mu\nu} - \ddot{\mathfrak{s}}^{\mu\nu} = \Theta^{\mu\nu} . \tag{6}$$

One now has to choose the expansion parameter: the most suitable one would be the reciprocal velocity of light, but it has already been taken as unity. An equivalent choice is to assume that the velocities are small quantities of the first order. Accelerations and masses are therefore of the second order. As the only dependence of the field quantities on time is through the positions and velocities of the sources, it follows that a time derivative of a field quantity is by one order of magnitude smaller than a space derivative.

One thus expands [12]

$$\mathfrak{h}^{\mu\nu} = 4\sum_{n=0}^{\infty}\left\{\left[\frac{1}{(2n)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\sum Mv^\mu v^\nu R^{2n-1}\right] + \varepsilon\left[\frac{1}{(2n+1)!}\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}}\sum Mv^\mu v^\nu R^{2n}\right]\right\} , \tag{7}$$

where ε is an arbitrary constant (for pure retarded potentials, $\varepsilon = -1$). Here $R = \sqrt{R^k R^k}$ and $R^k = x^k - \xi^k$.

[11] W. Tulczyjew: *Bull. Acad. Polon. Sc.*, **5**, 279 (1957).
[12] A. Peres: *Nuovo Cimento*, **11**, 617 (1959).

In this expression, we suppose that the co-ordinates ξ^k and velocities v^k are known (initial conditions) and that the masses and accelerations have to be computed

$$M = m + \underset{1}{m} + \underset{2}{m} + \dots ,$$

$$\dot{v}^k = a^k + \underset{1}{a}^k + \underset{2}{a}^k + \dots .$$

Note that $\underset{n}{m}$ and $\underset{n}{a}^k$ are small quantities of order $(n+2)$.

3. – Non-radiative approximations.

Let us now work step by step. The lowest order term in $\mathfrak{g}^{\mu\nu}$ is

$$\underset{2}{\mathfrak{g}}^{00} = \underset{2}{\mathfrak{h}}^{00} = 4 \sum (m/R) ,$$

Introducing this into (4) one gets the Newtonian approximation

$$\dot{m} = 0 , \qquad a^k = \frac{\partial}{\partial \xi^k} \sum{}' \frac{m}{R} .$$

Next, one has $\underset{3}{\mathfrak{g}}^{00} = \underset{3}{\mathfrak{g}}^{kl} = 0$, and $\underset{3}{\mathfrak{g}}^{0k} = \underset{3}{\mathfrak{h}}^{0k} = 4 \sum (mv^k/R)$, whence it follows that there is no first order correction to the Newtonian approximation $(\underset{1}{m} = \underset{1}{a}^k = 0)$ and that the second order correction to mass is [12]

$$\underset{2}{m} = \tfrac{1}{2} m v^2 + 3m \sum{}' (m/R) .$$

Non-linear contributions to the field appear at the fourth order [12]. One has

$$\begin{cases} \underset{4}{\mathfrak{S}}^{00} = 7 \sum \dfrac{m^2}{R^2} + 14 \sum\limits_{A,B}{}' \overset{A}{m} \overset{B}{m} S^{kk} , \\ \underset{4}{\mathfrak{S}}^{0k} = 0 , \\ \underset{4}{\mathfrak{S}}^{kl} = \dfrac{1}{2} \sum m^2 \left[\dfrac{\delta^{kl}}{R^2} - (\log R)_{,kl} \right] + \sum\limits_{A,B}{}' \overset{A}{m} \overset{B}{m} (2\delta^{kl} S^{nn} - 4 S^{kl}) , \end{cases}$$

where

$$S^{kl} \equiv \frac{\partial}{\partial \overset{A}{\xi}{}^k} \frac{\partial}{\partial \overset{B}{\xi}{}^l} \log (\overset{A}{R} + \overset{B}{R} + D) ,$$

and

$$D = \sqrt{D^k D^k} , \qquad D^k = \overset{A}{\xi}{}^k - \overset{B}{\xi}{}^k .$$

This has to be added to

$$\begin{cases} \underset{4}{\mathfrak{h}}^{00} = 4\sum\left(\frac{\overset{m}{2}}{R} + \frac{m}{2}\ddot{R}\right), \\ \underset{4}{\mathfrak{h}}^{0k} = 0\,, \\ \underset{4}{\mathfrak{h}}^{kl} = 4\sum\frac{mv^k v^l}{R}\,. \end{cases}$$

From this, we can compute the second order correction to the acceleration $\underset{2}{a}^k$ which leads, in the two-body problem, to the perihelion advance [8].

4. – Radiation field.

The radiation field—*i.e.* that part of $\mathfrak{g}^{\mu\nu}$ which is proportional to ε—appears in the fifth order. (Terms proportional to ε^2 appear only in the ninth order, and we shall not have to deal with them). From (7), one has

$$\underset{5}{\mathfrak{h}}^{00} = 4\varepsilon\sum\left[\dot{\underset{2}{m}} + \frac{m}{6}(\dddot{R}^2)\right],$$

$$\underset{5}{\mathfrak{h}}^{kl} = 4\varepsilon\sum m(\dot{v^k v^l})\,.$$

As $\underset{5}{\Theta}^{00} = \underset{5}{\Theta}^{kl} = 0$, one is tempted to write $\underset{5}{\mathfrak{F}}^{00} = \underset{5}{\mathfrak{F}}^{kl} = 0$. This is indeed what we have done in our previous paper (and what has also been done by other authors). As a matter of fact, the situation is not so simple; as clearly stated by SCHEIDEGGER [13], the problem is not to find non-conservative equations of motion, but to show that these solutions indeed correspond to *free* particles, *i.e.* to purely outgoing waves. Unless this is proved, it may always be objected that the energy change of the system, or part of it, is due to the interaction with some external radiation field.

The difficulty is essentially due to the non-linearity of the field equations: one cannot state that a given term, in the field, is due to a certain source, because the field does not depend linearly on the sources. There is therefore a danger of introducing into the solution some external field (pure radiation) that is not caused by the sources under consideration, but that influences their motion.

In practice, the difficulty appears in the following way: at each stage of the approximation procedure, one has to solve a Poisson equation, and there

[13] A. E. SCHEIDEGGER: *Phys. Rev.*, **99**, 1883 (1955).

is a considerable freedom of choice of solutions, each representing a possible motion and a gravitational field belonging thereto ([13]). Only one of these solutions behaves at infinity as purely outgoing waves ([9]); the remaining ones contain also incoming waves. It is in general difficult to determine which solution is the correct one, because the n-th term of a series expansion in powers of (v/c) behaves in the wave zone as R^{n-2}, and no boundary conditions for each stage of the procedure are known.

As far as $\mathfrak{h}^{\mu\nu}$ is concerned, the difficulty can be obviated by directly taking the expansion of the Liénard-Wiechert potentials, *i.e.* eq. (7). However, it is much more difficult to determine the radiative part of $\mathfrak{g}^{\mu\nu}$ ([14]). Some information can however be obtained by the following argument:

A solution behaving at infinity like $f(t-R/c)/R$ has an expansion $f(t)/R - f'(t)/c + \ldots$. It is therefore reasonable to stipulate that if a term such as $f(t)/R$ occurs at some approximation stage, one must add $-f'(t)$ — or, more generally, $\varepsilon f'(t)$ — at the next stage ([15]). (Note that this is a solution of the Laplace equation). In the case that $f(t)$ is a constant, this vanishes, but two approximations higher there will be a term $\varepsilon f \ddot{R}^2/6$. (Note that this also is solution of the Laplace equation.) For instance, $\underset{4}{\mathfrak{h}}{}^{kl} = 4\sum (m r^k v^l/R)$ is followed by $\underset{5}{\mathfrak{h}}{}^{kl} - 4\varepsilon \sum m(\dot{r}^k v^l)$, and $\underset{2}{\mathfrak{h}}{}^{00} - 4\sum (m/R)$ is followed, in the third place (*i.e.* in $\underset{5}{\mathfrak{h}}{}^{00}$) by $4\varepsilon \sum m\ddot{R}^2/6$.

Now it is easily shown that for large R

$$\mathfrak{g}^{kl} \to \frac{-\delta^{kl} + D^k D^l/D^2}{2DR},$$

so that

$$\underset{4}{\mathfrak{g}}{}^{00} \to -\frac{14}{R} \sum{}' \frac{\overset{A}{m}\overset{B}{m}}{D},$$

and

$$\underset{4}{\mathfrak{g}}{}^{kl} \to -\frac{2}{R} \sum{}' \overset{A}{m}\overset{B}{m} \frac{D^k D^l}{D^3}.$$

We thus have

$$\underset{5}{\mathfrak{g}}{}^{00} = -14\varepsilon \sum{}' \overset{A}{m}\overset{B}{m} \left(\frac{\dot{1}}{D}\right),$$

and

$$\underset{5}{\mathfrak{g}}{}^{kl} = -2\varepsilon \sum{}' \overset{A}{m}\overset{B}{m} \left(\frac{D^k D^l}{D^3}\right).$$

([14]) I am indebted to Dr. D. W. Sciama for calling my attention to this point.
([15]) A. Peres: *Nuovo Cimento*, **13**, 670 (1959).

Further one has (12)

$$\underset{5}{\mathfrak{H}}^{0k} = 4 \sum \left[\underset{2}{m} \frac{v^k}{R} + \frac{1}{2} m (v^k \ddot{R}) \right],$$

$$\underset{5}{\mathfrak{S}}^{0k} - \frac{1}{2} \sum m^2 \left[15 \frac{v^k}{R^2} - r^l (\log R)_{,kl} \right] + 16 \sum_{A,B}{}' \overset{A}{m} \overset{B}{m} \left(\frac{3}{4} \overset{A}{v}{}^l S^{lk} + \overset{B}{v}{}^k S^{ll} - \overset{B}{v}{}^l S^{lk} \right).$$

At large distances, this last expression behaves as

$$\underset{5}{\mathfrak{S}}^{0k} \to \frac{8}{R} \sum{}' \frac{\overset{A}{m} \overset{B}{m}}{D} \left(-\frac{3}{4} \overset{A}{v}{}^k - \overset{B}{v}{}^k + \frac{3}{4} \overset{A}{v}{}^l \frac{D^k D^l}{D^2} - \overset{B}{v}{}^l \frac{D^k D^l}{D^2} \right).$$

Taking into account that $\sum m a^k = 0$, one gets

$$\underset{6}{\mathfrak{S}}^{0k} = -\, 2\varepsilon \sum{}' \overset{A}{m} \overset{B}{m} \left(\frac{r^i \dot{D}^k D^l}{D^3} \right),$$

which has to be added to

$$\underset{6}{\mathfrak{H}}^{0k} = 4\varepsilon \sum \left[(\underset{2}{\dot{m}} \dot{v}^k) + \underset{2}{m} a^k + \frac{m}{6} (v^k \dddot{R}^2) \right].$$

(It is easily verified that $\underset{5}{\mathfrak{g}}^{00}{}_{,0} + \underset{6}{\mathfrak{g}}^{0k}{}_{,k} = 0$ as it should be. The difficulties that previously arose concerning this point were due to a different choice of the radiative fields in the two expressions.)

5. – Lorentz-invariance.

If it can be shown that the method developed in the previous sections is Lorentz-invariant, we can be assured that it gives the correct result because of Fock's uniqueness theorem (9).

We have supposed up to now that velocities were small quantities of the first order. This is equivalent to the assumption that the reciprocal velocity of light (which has been taken here as unity) is small of the first order. As the velocity of light is Lorentz-invariant *by definition*, our approximation method should also be Lorentz-invariant. This has to be understood in the following way:

Let us perform a Lorentz transformation

$$x'^k = x^k - \frac{\beta^k t}{\sqrt{1-\beta^2}} + \left[\frac{1}{\beta^2 \sqrt{1-\beta^2}} - \frac{1}{\beta^2} \right] \beta^k \beta^l x^l, \qquad t' = \frac{t - \beta^l x^l}{\sqrt{1-\beta^2}},$$

of all the quantities that appear in our equations. The velocity β^k is an arbitrary constant, subject only to the limitation of being small of the first order.

In general, $\underset{n}{\mathfrak{g}}'^{\mu\nu}$ will now depend on all the $\underset{m}{\mathfrak{g}}^{\mu\nu}$, $m \leqslant n$. *We require* $\underset{n}{\mathfrak{g}}'^{\mu\nu}$ *to depend in the same way on the transformed sources, as* $\underset{n}{\mathfrak{g}}^{\mu\nu}$ *on the original sources.*

This requirement may further limit the freedom of choice of solutions that still remains after the rules given in the previous section. In practice, it may be very difficult to apply it, because the R^k are not the spatial components of a four vector (they are defined for *equal times* of the source and field point). Their transformation law is therefore very complicated: it involves not only β^k and v^k, but also all higher time derivatives of v^k.

Fortunately, the rules of the previous section are sufficient to give unambiguous results up to the seventh order, and we shall have no need to test the Lorentz-invariance of our formulae. We shall take it for granted, because they seem to be the only reasonable solution of Lorentz-invariant equations.

6. – Radiation reaction.

We can already compute the fifth order radiative correction to the mass. From (4), one has $\underset{5}{\dot{m}} = m(\frac{3}{4}\underset{5}{\mathfrak{g}}^{00} - \frac{1}{4}\underset{5}{\mathfrak{g}}^{kk})_{,0}$. It can be shown, however, that $\underset{5}{\mathfrak{g}}^{kk} = 3\underset{5}{\mathfrak{g}}^{00}$, so that $\underset{5}{m} = 0$.

The fifth order radiative correction to the acceleration is given by (4), as

$$\underset{5}{a}^k = \underset{6}{\mathfrak{g}}^{0k}{}_{,0} + \tfrac{1}{4}\,\underset{7}{\mathfrak{g}}^{\mu\mu}{}_{,k} - \tfrac{1}{2}\underset{5}{\mathfrak{g}}^{00}\underset{2}{\mathfrak{g}}^{00}{}_{,k} - \tfrac{1}{4}\underset{5}{\mathfrak{g}}^{kl}\underset{2}{\mathfrak{g}}^{00}{}_{,l} - v^k\underset{5}{\mathfrak{g}}^{kl}{}_{,0}\,,$$

where $\mathfrak{g}^{\mu\mu} = \mathfrak{g}^{00} + \mathfrak{g}^{kk}$. We thus still need $\underset{7}{\mathfrak{g}}^{\mu\mu}{}_{,k}$. One has, from (7)

$$\left|\begin{array}{l} \underset{7}{\mathfrak{h}}^{00} = 4\varepsilon\sum\left[\dot{\underset{4}{m}} + \dfrac{1}{3\,!}\,(\dddot{\underset{2}{m}}\underset{2}{R}^2) + \dfrac{1}{5\,!}\,(\ddddot{m}R^4) + 2mv^k\underset{2}{a}^k - \dfrac{m}{3}\,R^k\dot{\underset{2}{\alpha}}^k\right], \\ \\ \underset{7}{\mathfrak{h}}^{kl} = 4\varepsilon\sum\left[m(v^k\underset{2}{a}^l + v^l\underset{2}{a}^k) + (\dot{\underset{2}{m}}v^kv^l) + \dfrac{m}{3\,!}\,(v^k\dddot{v}^lR^2)\right]. \end{array}\right.$$

Moreover, one has, from (6)

$$\left|\begin{array}{l} \nabla^2\underset{7}{\mathfrak{F}}^{00} = \underset{7}{\Theta}^{00} + \ddot{\underset{5}{\mathfrak{F}}}^{00} = \underset{5}{\mathfrak{g}}^{kl}\underset{2}{\mathfrak{g}}^{00}{}_{,kl} - 14\varepsilon\sum{}'\,\underset{m}{A}\,\underset{m}{B}\left(\dddot{\frac{1}{D}}\right), \\ \\ \nabla^2\underset{7}{\mathfrak{F}}^{kl} - \ddot{\underset{5}{\mathfrak{F}}}^{kl} = -2\varepsilon\sum{}'\,\underset{m}{A}\,\underset{m}{B}\left(\frac{\dddot{D}^k D^l}{D^3}\right), \end{array}\right.$$

whence

$$\nabla^2 \underset{7}{\mathfrak{g}}^{\mu\mu} = \underset{5}{\mathfrak{g}}^{kl}\, \underset{2}{\mathfrak{g}}^{00}{}_{,kl} - 16\varepsilon \sum{}' \overset{A}{m}\overset{B}{m} \left(\frac{\dddot{1}}{D}\right). \tag{8}$$

From the rules of Section **4**, we must also know the asymptotic behavior of $\underset{6}{\mathfrak{g}}^{\mu\mu}$ in order to solve (8). One has, from (6)

$$\nabla^2 \underset{6}{\mathfrak{g}}^{\mu\mu} = \underset{6}{\Theta}^{\mu\mu} + \frac{\partial^2}{\partial t^2}\,(\underset{4}{\mathfrak{g}}^{00} + \underset{4}{\mathfrak{g}}^{kk}) = \tag{9}$$

$$= \underset{6}{\Theta}^{\mu\mu} + 8\,\frac{\partial^2}{\partial t^2}\left\{\sum \frac{m^2}{R^2} + \sum_{A,B}{}' \left(\frac{1}{\overset{A}{R}\overset{B}{R}} - \frac{1}{D\overset{A}{R}} - \frac{1}{D\overset{B}{R}}\right)\right\}.$$

Here, $\underset{6}{\Theta}^{\mu\mu}$ is an extremely cumbersome expression. It is sufficient for our purpose to know that it falls off, at large distances, like R^{-4}, and therefore its contribution to $\underset{6}{\mathfrak{g}}^{\mu\mu}$ falls off as $f(t)/R$. This produces a term like $\varepsilon f'(t)$ in $\underset{7}{\mathfrak{g}}^{\mu\mu}$, which does not concern us, however, because only the spatial derivatives of this expression are needed. The curled bracket in (9) gives a contribution to $\underset{6}{\mathfrak{g}}^{\mu\mu}$:

$$8\,\frac{\partial^2}{\partial t^2}\left\{\sum m^2 \log R + \sum{}' \overset{A}{m}\overset{B}{m}\left[\log\,(\overset{A}{R} + \overset{B}{R} + D) - \frac{\overset{A}{R} + \overset{B}{R}}{D}\right]\right\}.$$

Unless all $\dot{D} = 0$, this expression behaves at large distances like R, and there corresponds to it, in $\underset{7}{\mathfrak{g}}^{\mu\mu}$, a term behaving as R^2. This can readily be verified from the solution of (8):

$$\underset{7}{\mathfrak{g}}^{\mu\mu} = 2\left(\underset{5}{\mathfrak{g}}^{ll} \sum \frac{m}{R} - \underset{5}{\mathfrak{g}}^{kl} \sum \frac{mR^k R^l}{R^3}\right) - 4\varepsilon \sum{}' \overset{A}{m}\overset{B}{m}\left(\frac{\overset{A}{R^2} + \dddot{} + \overset{B}{R^2}}{D}\right) + \varepsilon f'(t).$$

All the quantities needed to compute $\underset{5}{a}^k$ are thus known.

7. – Radiated energy.

We define the radiated energy, per unit time, as the rate of work of the particles against their own radiation field:

$$U = -\sum m\, \underset{5}{a}^k v^k.$$

For the sake of simplicity, we shall restrict ourselves to the case of two particles revolving on circular orbits (in the Newtonian approximation) at a distance D from each other. In this case, $\dot{\underset{2}{m}}$, $\underset{5}{\mathfrak{g}}^{00}$ adn $\underset{5}{\mathfrak{g}}^{kk}$ vanish. Next we choose, for convenience, a system of co-ordinates where the center of mass is at rest: $\sum m r^k = 0$. As a consequence, we can neglect all the spatial constants which

appear in the first four terms of $\underset{5}{a}^k$, as they give equal contributions to the accelerations of both particles, and will therefore cancel in the computation of U. Thus the only relevant term of $\underset{7}{\mathfrak{h}}^{\mu\mu}$ is $4\varepsilon \sum m\ddddot{R}^4/5!$ and the only relevant term of $-\underset{6}{\mathfrak{g}}^{0k}{}_{,0}$ is $-4\varepsilon \sum m(v^k\dddot{R}^2)/3!$ Moreover it can be shown [8] that the contribution from $\underset{7}{\xi}^{\mu\mu}$ exactly cancels the fourth term of $\underset{5}{a}^k$. One thus gets

$$\underset{5}{a}^k \approx -v^l\underset{5}{\mathfrak{g}}^{kl}{}_{,0} + \varepsilon \sum m\left[\frac{2}{3}(v^k\dddot{R}^2) + \frac{1}{30}(R^2\ddddot{R}^k)\right],$$

where the $\approx$ sign recalls that some spatial constants have been omitted, and where the right hand member has to be computed at the position of the particle under consideration. A straightforward computation gives

$$\underset{5}{a}^k \approx \frac{\varepsilon}{30}\frac{mv^k}{D^4}(m+M)(461\,M - 269\,m),$$

where M is the mass of the other particle. It follows that

$$U = -\varepsilon\frac{32}{5}\frac{m^2M^2(m+M)}{D^5}. \tag{10}$$

For purely outgoing waves ($\varepsilon = -1$), this agrees with the result of the linearized theory [1]. (The fact that one has previously obtained a negative radiated energy should be ascribed to the presence of incoming gravitational waves, which were absorbed by the particles).

The physical reality of this phenomenon can be confirmed by computing the rate of change of the total mass. One finds [16]

$$\sum \underset{7}{\dot{m}} = -U,$$

as one should expect.

8. – Behavior of the field at very large distances.

Henceforth we take, for the sake of simplicity, $m = M$. The evolution of the system is ruled by the equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(mv^2 - \dfrac{m^2}{D}\right) = -U = -\dfrac{64}{5}\dfrac{m^5}{D^5}, \\ \dfrac{2mv^2}{D} = \dfrac{m^2}{D^2}, \end{cases}$$

[16] A. PERES: *Nuovo Cimento*, **13**, 439 (1959).

whence

$$D^4 = \frac{512}{5} m^3 (T - t) . \tag{11}$$

T is the time at which radiative capture would occur, if the last equations were rigorously correct. Actually, all the previous theory is valid only for low velocities, *i.e.* as long as $D \gg m$. For very small D, capture occurs even if radiative effects are discarded [17].

From (10) and (11), one gets

$$U(t) = \frac{1}{80} \left(\frac{\frac{5}{2} m}{T - t} \right)^5 .$$

It follows that, at a distance R, the average energy flux and energy density of the gravitational waves are about

$$\bar{\sigma} = \frac{U(t - R)}{4\pi R^2} = \frac{1}{320\pi R^2} \left(\frac{\frac{5}{2} m}{T - t + R} \right)^5 . \tag{12}$$

For $R \ll T - t = 5D^4/512$ m^3, this varies as R^{-2}, but for very large R,

$$R > \frac{D^4}{m^3} ,$$

one has

$$\bar{\sigma} \approx \frac{(\frac{5}{2} m)^5}{320\pi} R^{-13/4} .$$

As $\bar{\sigma}$ is proportional to the square of the amplitude of the radiation field, it follows that the latter falls as $R^{-13/8}$.

On the other hand, the energy density $\bar{\sigma}$ produces an additional field [18], the potential of which varies, at large distances, like $R^{-5/4}$, *i.e.* more slowly than the amplitude of the radiation field. This additional field, however, is quasi-static, as shown by (12): its time and space derivatives behave like $R^{-9/4}$, *i.e.*, they fall off faster than those of the radiation field. Therefore the energy density of this additional field can be neglected with respect to $\bar{\sigma}$.

Let us summarize: at very large distances ($R > D^4/m^3$), the field contains a static part behaving as (m/R), a quasi-static part behaving as $(m/R)^{5/4}$ and a dynamic part behaving as $(m/R)^{13/8}$. (This result does not contradict a

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[17] C. Darwin: *Proc. Roy. Soc.*, A **249**, 180 (1959).

[18] A. Peres and N. Rosen: *Phys. Rev.* **115**, 1085 (1959).

theorem of Papapetrou ([19]) according to which the field can be asymptotically Euclidean at infinity only if it is a static both for $t \to +\infty$ and $t \to -\infty$, because the validity of this theorem is limited to the cases where there exists some constant radius R_0 such that space is empty for $R > R_0$ at all times t. In our case, however, it is easy to see from (11) that no such R_0 exists.)

The fact that the energy density of the gravitational field behaves at large distances like $R^{-13/4}$ has the important consequence that the total energy of the gravitational field is bounded. (It would not be bounded if the energy density behaved like R^{-2}, or even R^{-3}.) It is therefore possible to define a total energy of the particles and the field. This is constant, because the well-known conservation theorem

$$[\mathfrak{g}(\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu})]_{,\nu} = 0$$

can be written

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int \mathfrak{g}(\mathfrak{T}^{00} + \mathfrak{t}^{00})\,\mathrm{d}V = \int \mathfrak{g}\mathfrak{t}^{0k}\,\mathrm{d}S_k\,, \tag{13}$$

and the right hand member behaves like $R^{-5/4}$ for very large distances ($R > D^4/m^3$) and thus asymptotically vanishes at infinity. In fact, by considering the situation when the particles were very far apart, one gets

$$\int_{\text{all space}} \mathfrak{g}(\mathfrak{T}^{00} + \mathfrak{t}^{00})\,\mathrm{d}V = \sum m\,.$$

This fact was incorrectly interpreted by Infeld ([4]) as a proof that gravitational radiation does not exist. Actually, only the sum $\int \mathfrak{g}\mathfrak{T}^{00}\mathrm{d}V + \int \mathfrak{g}\mathfrak{t}^{00}\mathrm{d}V$ is constant, while energy is constantly pumped from the matter to the radiation field. Indeed, if we limit the domain of integration to the wave zone, *i.e.*, to distances such that

$$\left(\frac{D^3}{m}\right)^{\frac{1}{2}} \ll R \ll \frac{D^4}{m^3}\,, \tag{14}$$

both members of (13) are positive and approximately independent of R, as will be shown in the next section.

All the previous arguments are valid only in the case of retarded potentials (outgoing waves, $\varepsilon = -1$). In the case of half-retarded, half-advanced potentials (standing waves, $\varepsilon = 0$), no energy is radiated and the motion is truly periodic. The field then diverges logarithmically at spatial infinity ([18,20]).

([19]) A. Papapetrou: *Ann. Phys.*, **2**, 87 (1958).
([20]) A. Papapetrou: *Ann. Phys.*, **20**, 399 (1957); **1**, 186 (1958).

9. – Validity of the linear approximation.

We already know that the linear approximation is valid neither at very small distances from the sources (of the order of the Schwarzschild radius), nor at very large distances ($R > D^4/m^3$) where the quasi-static field caused by $\bar{\sigma}$ is more important than the radiation field. Its domain of validity, if it exists, must therefore be limited to the wave zone, defined by (14).

It is therefore important to examine in detail how the linearized theory can nevertheless correctly give the radiated energy. We here follow the proof of LANDAU and LIFSHITZ [1]. The « solution » of (1) is

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + 4\int \frac{\{\mathfrak{g}\mathfrak{T}^{\mu\nu} - (1/16\pi)\Theta^{\mu\nu}\}}{r}\,\mathrm{d}V,$$

where the curled bracket has to be computed at a retarded time. Let

$$\tau^{\mu\nu} = \mathfrak{g}\mathfrak{T}^{\mu\nu} - \frac{1}{16\pi}\Theta^{\mu\nu}.$$

For large R, one can write

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \frac{4}{R}\left\{\int \tau^{\mu\nu}\,\mathrm{d}V\right\}.$$

Moreover, it follows from (1) and the harmonic condition that $\tau^{\mu\nu}{}_{,\nu} = 0$, whence

$$\tau^{00}{}_{,0} = -\tau^{0k}{}_{,k}, \qquad \tau^{k0}{}_{,0} = -\tau^{kl}{}_{,l}.$$

Let us multiply the first of these equations by x^l and integrate over a large sphere of radius R. One gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \tau^{00}x^l\,\mathrm{d}V = -\int \tau^{0k}{}_{,k}x^l\,\mathrm{d}V = \int \tau^{0l}\mathrm{d}V - \int (\tau^{0k}x^l)_{,k}\,\mathrm{d}V. \tag{15}$$

We shall now show that the last integral can be neglected. One has

$$\int (\tau^{0k}x^l)_{,k}\,\mathrm{d}V = \int \tau^{0k}x^l\,\mathrm{d}S_k.$$

Now τ^{0k} varies as R^{-2} (in the wave zone) so that it seems that this expression diverges as R. This is not the case, however, as the following arguments shows:

it is possible to consider, instead of an infinite wave train, a short pulse of gravitational waves ([21]). One can take the surface integral at a very large distance, so that the pulse has still not reached it. It then vanishes, and the volume integral vanishes also. The other integrals in eq. (15), however, remain finite.

Let us note that this argument is correct only if there is a possibility of dealing with linear wave trains, *i.e.* those that can be arbitrarily decomposed into independent pulses. In reality, there is an interaction between the various parts of a wave train, due to the non-linearity of the Einsteins equations. The previous argument is therefore valid only inasmuch one can neglect the gravitational influence of the energy density of the waves. It is therefore correct in the wave zone, but not for $R > D^4/m^3$.

Thus, in the wave zone

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \tau^{00} x^l \,\mathrm{d}V = \int \tau^{0l}\,\mathrm{d}V.$$

In a similar fashion, one can show that

$$\int \tau^{kl}\,\mathrm{d}V = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \tau^{00} x^k x^l \,\mathrm{d}V,$$

whence

$$\mathfrak{g}^{kl} - \eta^{kl} = \frac{2}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \tau^{00} x^k x^l \,\mathrm{d}V = \frac{2}{R}\sum m\,(\ddot{\xi^k}\xi^l)\,. \tag{16}$$

This is indeed equal to the limit of $\mathfrak{g}^{kl}_4$, for large R, as found in Section **3**. The only difference is that the right hand member is now to be computed at time $(t - R)$, rather than t. (Henceforth, it will always be understood that sources have to be considered at a retarded time.)

10. – The wave zone.

The wave zone is characterized, in the case of outgoing waves, by

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \approx \frac{\partial}{r\,\partial\theta} \approx \frac{\partial}{r\,\partial\varphi} \approx 0\,(r^{-2})\,.$$

Let

$$-n^k = n_k = R^k/R\,, \qquad -n^0 = -n_0 = 1\,,$$

([21]) D. Geissler, A. Papapetrou and H. Treder: *Ann. Phys.*, **2**, 344 (1959).

whence

$$n^\mu n_\mu = 0 \,. \tag{17}$$

One can write [22]

$$g^{\mu\nu}{}_{,\sigma} = a^{\mu\nu} n_\sigma / R \,, \tag{18}$$

where $a^{\mu\nu}$ may depend on n_k, but is almost independent of R (except through the retardation of time).

The harmonic condition now reads

$$a^{\mu\nu} n_\nu = 0 \,, \tag{19}$$

whence

$$a^{0k} = a^{kl} n_l \,, \tag{20}$$

and

$$a^{00} = a^{0k} n_k = a^{kl} n_k n_l \,. \tag{21}$$

The energy-momentum pseudotensor of the gravitational field is given by [18]

$$\begin{aligned} 16\pi g t^{\mu\nu} = g^{\alpha\beta}{}_{,\gamma} g^{\pi\varrho}{}_{,\tau} \big[& g^{\mu\nu} \left(\tfrac{1}{2} \delta^\gamma_\pi \delta^\tau_\alpha g_{\beta\varrho} - \tfrac{1}{4} g^{\gamma\tau} g_{\alpha\pi} g_{\beta\varrho} + \tfrac{1}{8} g^{\gamma\tau} g_{\alpha\beta} g_{\pi\varrho} \right) + \\ & + \delta^\mu_\alpha \delta^\nu_\pi g^{\gamma\tau} g_{\beta\varrho} + \delta^\mu_\alpha \delta^\nu_\beta \delta^\gamma_\pi \delta^\tau_\varrho - \delta^\mu_\alpha \delta^\nu_\pi \delta^\gamma_\beta \delta^\tau_\varrho - \delta^\mu_\alpha \delta^\gamma_\pi g^{\nu\tau} g_{\beta\varrho} - \\ & - \delta^\nu_\pi \delta^\tau_\alpha g^{\mu\gamma} g_{\beta\varrho} + \tfrac{1}{2} g^{\mu\gamma} g^{\nu\tau} g_{\alpha\pi} g_{\beta\varrho} - \tfrac{1}{4} g^{\mu\gamma} g^{\nu\tau} g_{\alpha\beta} g_{\pi\varrho} \big] \,. \end{aligned}$$

With the help of (17) and (18), one gets, in the linear approximation

$$t^{\mu\nu} = \sigma n^\mu n^\nu \,,$$

where

$$\sigma = \frac{1}{64\pi R^2} \left(2\eta_{\alpha\pi}\eta_{\beta\varrho} - \eta_{\alpha\beta}\eta_{\pi\varrho} \right) a^{\alpha\beta} a^{\pi\varrho} \,. \tag{22}$$

Introducing

$$b^{kl} = a^{kl} - \tfrac{1}{3} \delta^{kl} a^{nn} \,,$$

one gets, with the help of (20) and (21)

$$\sigma = \frac{1}{64\pi R^2} \left(2 b^{kl} b^{kl} - 4 b^{kp} b^{kq} n_p n_q + b^{kl} b^{pq} n_k n_l n_p n_q \right) .$$

[22] A. TRAUTMAN: *Bull. Acad. Polon. Sc.*, **6**, 407 (1958).

Moreover, one has [23]

$$\overline{n_p n_q} = \tfrac{1}{3}\delta_{pq}\,,$$

$$\overline{n_k n_l n_p n_q} = \frac{1}{15}\left(\delta_{kl}\delta_{pq} + \delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}\right),$$

whence

$$\bar{\sigma} = \frac{1}{80\pi R^2}\, b^{kl} b^{kl}\,.$$

This is always a positive quantity. It follows

$$U = \int t^{0k}\,\mathrm{d}S_k = \int \sigma n_k\,\mathrm{d}S_k = 4\pi R^2 \bar{\sigma} = \frac{1}{20}\, b^{kl} b^{kl}\,.$$

In the case of free particles, one has, from (16) and (18)

$$a^{kl} = -2\sum m(\dddot{\xi}^k \xi^l)\,, \tag{23}$$

whence

$$U = \frac{1}{45}\left[\sum m\left(3\xi^k\xi^l - \delta^{kl}\xi^n\xi^n\right)\right]^2 .$$

This is the usual formula for gravitational radiation [1]. As it has been *independently* checked both for free particles (Section 7) and for constrained motion [21], we can say that *the existence of gravitational radiation is now well established.*

From (20), (21) and (23), it follows that the form of g^{00} and g^{0k}, in the wave zone, is [25]:

$$g^{00} - 1 + \frac{4}{R}\sum m + \frac{2}{R}\, n_k n_l \sum m\,(\ddot{\xi}^k \xi^l)\,,$$

$$g^{0k} = \frac{4}{R}\sum m v^k + \frac{2}{R}\, n_l \sum m(\ddot{\xi}^k \xi^l)\,.$$

When we compare these formulae with the asymptotic values of $\underset{4}{g}^{00}$ and $\underset{5}{g}^{0k}$ that are given in Section 4 we find that they do not agree. This fact should not surprise us, because we have used in the first sections of this paper an

[23] L. Landau and E. Lifshitz: loc. cit., p. 206.

[24] W. B. Bonnor: *Nature*, **181**, 1196 (1958); *Phil. Trans. Roy. Soc.*, A **251**, 233 (1959); A. Peres and N. Rosen: *Ann. Phys.* (to be published).

[25] V. A. Fock: loc. cit., p. 417.

approximation method based on the assumption that time derivatives are much smaller than space derivatives. This is indeed correct in the vicinity of the particles, but in the wave zone, time derivatives are of the same order as radial space derivatives, and much larger than tangential space derivatives. The various orders of approximation are therefore completely different: quantities of the n-th order in one region contain also contributions from all lower orders in the other region, and no direct comparison can be made. The agreement for $\mathfrak{g}^{kl}$, that we have found in the previous section, should be ascribed to the fact that in both methods we were dealing with the lowest approximation to $\mathfrak{g}^{kl}$.

11. – Uniqueness of σ.

FOCK has shown (9) that the harmonic system of co-ordinates behaving at infinity as pure outgoing waves is unique (up to a Lorentz transformation). We have found, however, that the dominant term at very large distances has not the form of outgoing waves.

We now intend to show that it is sufficient to require that the system be harmonic and behave as outgoing waves in the wave zone (but not necessarily near the sources nor at infinity) in order to get a unique value for σ.

We thus assume only (18) and (19). Under an infinitesimal co-ordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$, the $\mathfrak{g}^{\mu\nu}$ transform according to

$$\mathfrak{g}'^{\mu\nu} = \mathfrak{g}^{\mu\nu} + \xi^{\mu,\nu} + \xi^{\nu,\mu} - \eta^{\mu\nu}\xi^{\sigma}{}_{,\sigma} .$$

As the $\mathfrak{g}'^{\mu\nu}$ are also to behave as outgoing waves, one must have

$$\xi^{\alpha}{}_{,\beta} = \frac{c^{\alpha} n_{\beta}}{R} ,$$

where the c^{α} themselves satisfy

$$c^{\alpha}{}_{,\beta} = f^{\alpha} n_{\beta} .$$

Thus

$$a'^{\mu\nu} = a^{\mu\nu} + f^{\mu} n^{\nu} + f^{\nu} n^{\mu} - \eta^{\mu\nu} f^{\sigma} n_{\sigma} . \tag{24}$$

It follows that $a'^{\mu\nu} n_{\nu} = 0$, *i.e.* the new system is also harmonic.

Let us now go back to eq. (22) which defines σ. A straightforward computation shows that

$$\sigma' = \frac{1}{64\pi R^2} (2\eta_{\alpha\pi}\eta_{\beta\varrho} - \eta_{\alpha\beta}\eta_{\pi\varrho}) a'^{\alpha\beta} a'^{\pi\varrho} ,$$

is exactly equal to σ. This could be foreseen by noting that (24) can be written as

$$a'^{\mu\nu} = a^{\mu\nu} + c^{\mu,\nu} + c^{\nu,\mu} - \eta^{\mu\nu} c^{\sigma}{}_{,\sigma} ,$$

i.e. $a_{\mu\nu}$ behaves as a tensor density would under a transformation $x'^{\mu} = x^{\mu} + c^{\mu}$.

One may also define σ with the help of the Einstein pseudotensor [26]

$$\mathfrak{t}_{\nu}^{\mu} = \frac{1}{16\pi} \left(\delta_{\nu}^{\mu} \mathfrak{L} - \mathfrak{g}^{\alpha\beta}{}_{,\nu} \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\alpha\beta}{}_{,\mu}} \right) ,$$

where

$$\mathfrak{L} = \mathfrak{g}^{\alpha\beta}{}_{,\gamma} \mathfrak{g}^{\pi\varrho}{}_{,\tau} \left(\frac{1}{2} \delta_{\varrho}^{\gamma} \delta_{\beta}^{\tau} \mathfrak{g}_{\alpha\pi} - \frac{1}{4} \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\alpha\pi} \mathfrak{g}_{\beta\varrho} + \frac{1}{8} \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\alpha\beta} \mathfrak{g}_{\pi\varrho} \right) .$$

Now

$$\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\alpha\beta}{}_{,\mu}} = -\frac{1}{2} \mathfrak{g}^{\pi\varrho}{}_{,\tau} \left[\mathfrak{g}^{\mu\tau} \left(\mathfrak{g}_{\alpha\pi} \mathfrak{g}_{\beta\varrho} - \frac{1}{2} \mathfrak{g}_{\alpha\beta} \mathfrak{g}_{\pi\varrho} \right) - \delta_{\alpha}^{\tau} \delta_{\pi}^{\mu} \mathfrak{g}_{\beta\varrho} - \delta_{\beta}^{\tau} \delta_{\pi}^{\mu} \mathfrak{g}_{\alpha\varrho} \right] ,$$

whence

$$\mathfrak{t}_{\nu}^{\mu} = \frac{1}{16\pi} \mathfrak{g}^{\alpha\beta}{}_{,\gamma} \mathfrak{g}^{\pi\varrho}{}_{,\tau} \left[\delta_{\nu}^{\mu} \left(\frac{1}{2} \delta_{\varrho}^{\gamma} \delta_{\beta}^{\tau} \mathfrak{g}_{\alpha\pi} - \frac{1}{4} \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\alpha\pi} \mathfrak{g}_{\beta\varrho} + \frac{1}{8} \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\alpha\beta} \mathfrak{g}_{\pi\varrho} \right) + \right.$$

$$\left. + \frac{1}{2} \delta_{\nu}^{\gamma} \left(\mathfrak{g}^{\mu\tau} \mathfrak{g}_{\alpha\pi} \mathfrak{g}_{\beta\varrho} - \frac{1}{2} \mathfrak{g}^{\mu\tau} \mathfrak{g}_{\alpha\beta} \mathfrak{g}_{\pi\varrho} - \delta_{\alpha}^{\tau} \delta_{\pi}^{\mu} \mathfrak{g}_{\beta\varrho} - \delta_{\beta}^{\tau} \delta_{\pi}^{\mu} \mathfrak{g}_{\alpha\varrho} \right) \right] .$$

With the help of (17) and (18), one gets, in the linear approximation

$$\mathfrak{t}_{\nu}^{\mu} = \frac{1}{64\pi R^2} n^{\mu} n_{\nu} (2\eta_{\alpha\pi}\eta_{\beta\varrho} - \eta_{\alpha\beta}\eta_{\pi\varrho}) a^{\alpha\beta} a^{\pi\varrho} .$$

We thus get exactly the same result as with the $\mathfrak{t}^{\mu\nu}$ of Landau and Lifshitz.

At last, let us note that a very simple *exact* formula for σ can be obtained from

$$16\pi g \mathfrak{t}^{00} = (\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{00} - \mathfrak{g}^{0\alpha}\mathfrak{g}^{0\beta})_{,\alpha\beta} ,$$

which holds *in vacuo* ($\mathfrak{T}^{00} = 0$). This can be written

$$16\pi g \mathfrak{t}^{00} = \left[(-g) g^{00} \left(g^{kl} - \frac{g^{0k} g^{0l}}{g^{00}} \right) \right]_{,kl} . \tag{25}$$

[26] R. C. Tolman: *Relativity Thermodynamics and Cosmology* (Oxford, 1934), p. 224.
[27] P. A. M. Dirac: *Proc. Roy. Soc.*, A **246**, 333 (1958).

However

$$\left(g^{kl} - \frac{g^{0k}g^{0l}}{g^{00}}\right) g_{lm} \equiv \delta^l_m ,$$

and

$$(-g)g^{00} \equiv -\,\mathrm{Det}\,(g_{kl}) .$$

Therefore the square bracket in (25) is simply minus the co-factor of g_{kl} in the determinant of the g_{mn}:

$$16\pi \mathfrak{g}\mathfrak{t}^{00} = -\tfrac{1}{2}\eta^{ijk}\eta^{lmn}(g_{jm}g_{kn})_{,il} .$$

One thus sees that only the six g_{kl} give a contribution to the energy density of the gravitational field. This is related to the fact that they are canonical variables, while the four $g_{0\mu}$ are not [27].

* * *

I am greatly indebted to Professor N. Rosen for many stimulating discussions and much helpful criticism.

RIASSUNTO (*)

Si espone un metodo ad approssimazioni successive per risolvere le equazioni di Einstein per un sistema di particelle polari che gravitano liberamente. Si dimostra che si può scegliere la soluzione che rappresenta soltanto onde uscenti. Si trova che la correzione di 5° ordine all'accelerazione comporta un termine non conservativo: l'energia si perde, per radiazione gravitazionale, in quantità esattamente uguale a quella predetta della teoria linearizzata. Questo può essere dimostrato calcolando la perdita di massa del sistema. Poi si passa ad esaminare la validità della teoria linearizzata: si dimostra che non può descrivere correttamente il campo a distanze molto grandi dall'origine, ma ciononostante da un risultato corretto per l'energia irradiata.

(*) *Traduzione a cura della Redazione.*

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On the Matrix Formulation of the Theory of Partial Polarization in Terms of Observables.

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Summary. — The coherency matrix of a quasi-monochromatic plane wave is deduced from a matrix representation of the analytic signal associated with the electric field. It is shown that if the radiation passes through a physical device, such as a compensator, absorber, rotator, or polarizer, the effect of this interaction can be fully described in terms of appropriately choosen operators which transform directly the coherency matrix. The complex degree of coherence is defined in terms of the operators mentioned above, and from this is deduced an expression characterizing the degree of polarization. It is shown that the quantity deduced in this manner is identical with that obtained from the more conventional definition. An experiment is described which can serve to measure the components of the correlation matrix.

1. – Introduction.

The study of partial polarization has a great deal in common with the study of partial coherence, since both are necessitated by the statistical nature of « natural » radiation. Thus for example a rigorously monochromatic beam of electromagnetic radiation would be completely coherent and completely polarized; and the behaviour of such a beam is simply determined. However, if one considers a more realistic model of a beam of light, *e.g.* quasi-monochro-

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matic (which results from the superposition of a large number of randomly timed statistically independent pulses with the same central frequency), the problem of describing its behaviour is essentially different. What was a completely determined problem becomes a statistical one in which even the concepts of amplitude and phase become ambiguous, requiring redefinition.

It is only when dealing with such statistical radiation that the terms partial coherence and partial polarization have any meaning.

In view of this genetic relationship between these two aspects of the study of natural light one would expect the techniques and concepts which prove important for one to be helpful in understanding the other. Consequently, the recent emphasis on the formulation of coherence theory in terms of measurable quantities (correlation functions), and the success which this approach has enjoyed, suggest that the more general problems of partial polarization might also be better understood if they were formulated in a similar way.

In fact, considerable progress with this approach has already been made through the introduction by WOLF [1] of the correlation tensor. The matrix of this tensor, the correlation matrix, is the appropriate entity for the description of partially polarized beams. Its physical significance was, however, left somewhat obscure. In a later paper WOLF [2] identified the elements of the coherency matrix $\mathscr{J}$ with the factors occurring in the expression for the intensity in a beam after transmission through a compensator and a polarizer. In his paper ref. [1] he pointed out that $\mathscr{J}$ is formally equivalent to the density matrix in the study of scattering phenomena.

In spite of the attention focussed upon the coherency matrix, the uniformity which this entity introduces has gone largely unnoticed. The formulation of the subject as introduced here stresses the analogy between this field of research and modern scattering theory.

We shall use here a matrix representation of the electric field and derive formulae for the solution of the problem arising in the description of partially polarized fields and their interaction with physical devices. We mention that in the last decade several attempts have been made to describe in terms of matrices the action of various physical devices on radiation passed through them (see, for example, CLARK JONES [3] or WESTFOLD [4], where also further literature is quoted). Our approach will be, however, very different from these and also more general. In particular, we shall avoid the use of the «Stokes vector»

$$\boldsymbol{s} = (s_0, s_1, s_2, s_3) ,$$

(1) E. WOLF: *Nuovo Cimento*, **12**, 884 (1954).
(2) E. WOLF: *Nuovo Cimento*, **13**, 1165 (1959).
(3) R. CLARK JONES: *Journ. Opt. Soc. Am.*, **46**, 126 (1956).
(4) K. C. WESTFOLD: *Journ. Opt. Soc. Am.*, **49**, 717 (1959).

which, in fact, has not the transformation property of a four-vector. Furthermore, we shall not need 4×4 matrices to characterize the interactions, but only 2×2 matrices. An attractive feature of the present approach is that *one obtains matrix operators characteristic of the physical devices (interactions) which operate directly on the coherency matrix.* Further, the physical characteristic to be measured (*e.g.* the intensity I) is always given by an expression of the form

$$F = \mathrm{Sp}\,[\mathscr{F}\,\mathscr{I}]\,, \tag{1}$$

where F is the characteristic to be measured, $\mathscr{I}$ is the coherency matrix, $\mathscr{F}$ is an operator describing the experiment (the device), and Sp indicates the trace. In (1) *the operator $\mathscr{F}$ depends only on the physical devices and $\mathscr{I}$ depends only on the measurable properties of the beam.* That is, $\mathscr{I}$ describes the state of the field.

2. – Scope, definitions and notation.

Throughout this discussion we shall limit our attention to quasi-monochromatic fields. In essence this approximation requires that the spectral width of the radiation is negligible compared to the mean frequency, *i.e.*

$$\Delta\nu \ll \bar{\nu}\,. \tag{2a}$$

The introduction of this approximation enables us to evaluate frequency-dependent quantities at the mean frequency, but also limits the validity of the development to phenomena involving relatively small path differences Δl, *i.e.* the theory is valid when

$$\Delta l \ll c/\Delta\nu\,, \tag{2b}$$

where c is the velocity of light. For a full discussion of the consequences of this approximation the reader is referred to Born and Wolf [5]. Further, we limit our attention to plane waves in which case the electric vector has two components, perpendicular to the direction of propagation. While these approximations are appropriate for many problems of interest, the analysis given here may be extended to polychromatic non-planar waves. This generalization will be treated in a later communication.

[5] M. Born and E. Wolf: *Principles of Optics* (London, 1959), p. 502.

The real electric field will be denoted by the column matrix

$$\mathcal{E}^r(\boldsymbol{x}, t) = \begin{pmatrix} E_x^r(\boldsymbol{x}, t) \\ E_y^r(\boldsymbol{x}, t) \end{pmatrix}. \tag{3}$$

Here, as throughout this paper, script letters denote matrices; bold face type denotes vectors; and a superscript r denotes a real function of a real variable.

Our analysis will not deal directly with $\mathcal{E}^r$ but rather with the associated analytic signal representation of the field. The advantage of this representation for problems involving correlation functions become apparent from the investigations of various authors ([6-8]); therefore we shall simply review briefly the method of obtaining the analytic signal from the real field.

It is assumed (see p. 1169 of ref. ([2])) that the field possesses a Fourier transform; thus

$$E_x^r(\boldsymbol{x}, t) = \int_0^\infty a(\boldsymbol{x}, \nu) \sin [\varphi(\boldsymbol{x}, \nu) - 2\pi\nu t] \, \mathrm{d}\nu .$$

We associate with E_x^r its « conjugate function » $E_x^i(\boldsymbol{x}, t)$ defined by

$$E_x^i(\boldsymbol{x}, t) = \int_0^\infty a(\boldsymbol{x}, \nu) \cos [\varphi(\boldsymbol{x}, \nu) - 2\pi\nu t] \, \mathrm{d}\nu .$$

It is easily shown that E_x^i is the Hilbert transform of E_x^r, *i.e.*

$$E_x^i(\boldsymbol{x}, t) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{E_x^r(\boldsymbol{x}, t')}{t' - t} \, \mathrm{d}t' ,$$

where P denotes Cauchy's principal value. The analytic signal E_x is then defined as

$$E_x(\boldsymbol{x}, t) = E_x^r(\boldsymbol{x}, t) + iE_x^i(\boldsymbol{x}, t) .$$

Similar considerations hold for E_y.

([6]) Cf. ref. ([5]), p. 492.
([7]) G. B. Parrent jr: *Journ. Opt. Soc. Am.*, **49**, 787 (1959).
([8]) P. Roman and E. Wolf: *Ann. Phys.*, in press.

In terms of these functions our representation becomes

$$\mathscr{E}(\boldsymbol{x}, t) = \begin{pmatrix} E_x(\boldsymbol{x}, t) \\ E_y(\boldsymbol{x}, t) \end{pmatrix} = \mathscr{E}^r(\boldsymbol{x}, t) + i\mathscr{E}^i(\boldsymbol{x}, t) \,. \tag{4}$$

In this representation the intensity I at the point $\boldsymbol{x}$ may be expressed as

$$I = \mathrm{Sp}\, \langle \mathscr{E} \times \mathscr{E}^\dagger \rangle \,. \tag{5}$$

Here $\mathscr{E}^\dagger$ is the Hermitian conjugate of $\mathscr{E}$, *i.e.* the row matrix

$$\mathscr{E}^\dagger = \tilde{\mathscr{E}}^* = [E_x^* \, E_y^*];$$

further, $\langle \ldots \rangle$ indicates the time average, and $\times$ denotes the Kronecker product of matrices. That this definition is equivalent to that normally given, may be seen by writing (5) in full, when we obtain

$$I = \langle \boldsymbol{E} \cdot \boldsymbol{E}^* \rangle = 2\langle (\boldsymbol{E}^r)^2 \rangle \,.$$

(In the last step some simple properties of Hilbert transforms have been utilized.)

We introduce now the coherency matriw $\mathscr{I}$ by the definition

$$\mathscr{I} = \langle \mathscr{E} \times \mathscr{E}^\dagger \rangle \,, \tag{6}$$

which is clearly Hermitian:

$$\mathscr{I}^\dagger = \mathscr{I} \,.$$

$\mathscr{I}$ is in fact the matrix of the correlation tensor introduced by WOLF [1] who proved that the elements of this matrix are observables of the radiation field. In terms of $\mathscr{I}$ the intensity (5) becomes simply

$$I = \mathrm{Sp}\, \mathscr{I} \,. \tag{7}$$

One further remark about notation. At one stage in the analysis the Pauli matrices will be used. Since our representation differs slightly from that normally used, we shall summarize it at this point. These matrices obey the algebra

$$\begin{cases} \sigma_\alpha \sigma_\beta = -i\sigma_\gamma \,, & (\alpha, \beta, \gamma) = (1, 2, 3) \text{ and cycl.}; \\ (\sigma_i)^2 = \sigma_0 \,, & (i = 0, 1, 2, 3) \,, \\ \sigma_i \sigma_0 = \sigma_0 \sigma_i = \sigma_i \,. \end{cases} \tag{8}$$

It follows from (8) that

$$\mathrm{Sp}\,(\sigma_i\sigma_j) = 2\delta_{ij}\,, \qquad (i, j = 0, 1, 2, 3)\,. \tag{9}$$

The matrices may be represented by putting

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{10}$$

In the following sections we shall obtain transformation equations for $\mathscr{I}$ corresponding to the passage of partially polarized radiation through the various physical devices which are of interest in the study of such fields. In Sections **8** and **9** we discuss the various parameters used to specify the state of such statistical radiation and their relation to the coherency matrix. The relation between $\mathscr{I}$ and the density matrix will be briefly discussed in the concluding remarks.

3. – Compensator.

The compensator is a device which introduces a phase change ε_x in the x-component and ε_y in the y-component in each spectral component of the field vector. This results in a relative phase difference $\delta = \varepsilon_x - \varepsilon_y$ between the x- and y-components. Thus the compensator produces a selective rotation of the electric field in Fourier space. In the general case δ is of course a function of frequency; however, consistent with the quasi-monochromatic approximation we take the phase shift for each spectral component to be equal to that for the mean frequency. The compensator may, therefore, be represented or characterized by a unitary rotation matrix

$$\mathscr{C} = \begin{pmatrix} \exp\left[\frac{i}{2}\delta\right] & 0 \\ 0 & \exp\left[-\frac{i}{2}\delta\right] \end{pmatrix}. \tag{11}$$

If $\mathscr{E}$ is the field incident on the compensator, the emergent field $\mathscr{E}_c$ may be written as

$$\mathscr{E}_c = \mathscr{C}\mathscr{E}; \tag{12}$$

and the coherency matrix for the emergent field is given, according to the definition (6), by

$$\mathscr{I}_c = \langle \mathscr{E}_c \times \mathscr{E}_c^{\dagger} \rangle = \langle \mathscr{C}\mathscr{E} \times \mathscr{E}^{\dagger}\mathscr{C}^{\dagger} \rangle\,. \tag{13}$$

Hence we may write (*)

$$\mathscr{I}_c = \mathscr{C}\langle \mathscr{E} \times \mathscr{E}^\dagger \rangle \mathscr{C}^\dagger ; \tag{14}$$

and substituting from (6) we obtain, using also the unitary nature of $\mathscr{C}$,

$$\mathscr{I}_c = \mathscr{C}\mathscr{I}\mathscr{C}^\dagger = \mathscr{C}\mathscr{I}\mathscr{C}^{-1} . \tag{15}$$

Thus the effect of the compensator may be characterized by a rotation operator in Fourier space acting directly on the coherency matrix $\mathscr{I}$, without recourse to the unmeasurable quantities of the $\mathscr{E}$ field itself.

To compute the intensity I_c at a point in the emergent beam we take, according to (7), the trace on both sides of (15); and since the argument of the trace may be cyclically permuted, we obtain

$$I_c = I .$$

That the intensity is unchanged under the passage of the beam through a compensator, could have been anticipated from the fact that a rotation leaves the trace invariant and absorption and reflection were ignored. However, absorption is easily incorporated into the scheme as will be shown in the next section.

4. – Absorption.

Absorption is characterized by a decrease in field strength and may therefore be represented by a matrix of the form

$$\mathscr{A} = \begin{pmatrix} \exp\left[-\tfrac{1}{2}\eta_x\right] & 0 \\ 0 & \exp\left[-\tfrac{1}{2}\eta_y\right] \end{pmatrix} . \tag{16}$$

Here η_x and η_y are the absorption coefficients for the x- and y-components respectively, evaluated at the mean frequency $\bar{\nu}$. The effect of absorption is to produce a field $\mathscr{E}_A$ given by

$$\mathscr{E}_A = \mathscr{A}\mathscr{E} , \tag{17}$$

(*) The associative property used in obtaining (14) is of course not permissible in general; indeed in the general case this operation is meaningless. However, when it is possible to perform such an operation, as in the present and in the following cases, the associative property is valid as may be easily verified.

and the coherency matrix becomes

$$\mathscr{J}_A = \mathscr{A}\mathscr{J}\mathscr{A}^\dagger . \tag{18}$$

The reduced intensity is clearly

$$I_A = \mathrm{Sp}\,[\mathscr{A}\mathscr{J}\mathscr{A}^\dagger] = \mathrm{Sp}\,[\mathscr{A}^2\mathscr{J}] . \tag{19}$$

Note that the measurable intensity is given by an expression of the form (1).

5. – Rotator.

Various materials and physical devices produce a rotation of the electric vector, *i.e.* they rotate the plane of polarization. Such a device is termed a rotator and may clearly be characterized by a rotation operator

$$\mathscr{R}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \tag{20}$$

where α is the angle through which the field is rotated. Note that $\mathscr{R}$ is a real antisymmetric unimodular unitary matrix.

The emergent field $\mathscr{E}_R$ following a rotator is given by

$$\mathscr{E}_R = \mathscr{R}(\alpha)\mathscr{E} , \tag{21}$$

and the coherency matrix becomes

$$\mathscr{J}_R = \mathscr{R}(\alpha)\mathscr{J}\mathscr{R}^\dagger(\alpha) = \mathscr{R}(\alpha)\mathscr{J}\mathscr{R}^{-1}(\alpha) = \mathscr{R}(\alpha)\mathscr{J}\mathscr{R}(-\alpha) . \tag{22}$$

Thus the action of the rotator may also be represented as an operation directly on the coherency matrix which contains only observable quantities. The intensity of course remains unchanged.

6. – Polarizer.

The last device to be considered in this discussion before turning to general considerations is a polarizer, such as a Nichol prism, which passes only a particular component of the field, say the component making an angle θ with the x-direction. That is, the polarizer takes the projection of the $\mathscr{E}$ field

on the direction θ. The polarizer may thus be characterized by a projection operator $\mathscr{P}(\theta)$. Projection operators are singular and satisfy the idempotency condition

$$\mathscr{P}_+(\theta)\mathscr{P}_+(\theta) = \mathscr{P}_+(\theta)\,. \tag{23}$$

Associated with every projection operator $\mathscr{P}_+(\theta)$ is an orthogonal proejction operator $\mathscr{P}_-(\theta)$ representing a projection on the direction orthogonal to θ and satisfying the conditions

$$\begin{cases} \mathscr{P}_-(\theta)\mathscr{P}_-(\theta) = \mathscr{P}_-(\theta)\,, \\ \mathscr{P}_+(\theta)\mathscr{P}_-(\theta) = \mathscr{P}_-(\theta)\mathscr{P}_+(\theta) = 0\,, \\ \mathscr{P}_+(\theta) + \mathscr{P}_-(\theta) = 1\,. \end{cases} \tag{24}$$

Since the polarizer takes a projection of the $\mathscr{E}$ field, it may be represented by the operator

$$\mathscr{P}_+(\theta) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{25}$$

The associated operator $\mathscr{P}_-(\theta)$ may be written as

$$\mathscr{P}_-(\theta) = \mathscr{P}_+\left(\theta + \frac{\pi}{2}\right) = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}. \tag{26}$$

That these Hermitian operators satisfy the above conditions (23) and (24) is readily verified.

The field $\mathscr{E}_P$ emerging from the polarizer is given as

$$\mathscr{E}_P = \mathscr{P}_+(\theta)\mathscr{E}\,; \tag{27}$$

and the coherency matrix becomes

$$\mathscr{J}_P = \mathscr{P}_+\mathscr{J}\mathscr{P}_+^\dagger = \mathscr{P}_+\mathscr{J}\mathscr{P}_+\,, \tag{28}$$

since $\mathscr{P}$ is Hermitian. Thus the effect of the polarizer is also represented by an operation directly on the observable entity $\mathscr{J}$. Unlike the devices previously considered, however, the polarizer does not leave the intensity unaltered. The intensity is given by

$$I_P = \mathrm{Sp}\,\mathscr{J}_P = \mathrm{Sp}[\mathscr{P}_+\mathscr{J}\mathscr{P}_+] = \mathrm{Sp}[\mathscr{P}_+\mathscr{J}]\,, \tag{29}$$

where the property (23) of $\mathscr{P}_-$ was used. The reduced intensity is again an expression of the form (1).

For further application we note here that, as follows directly from (25) and (20),

$$\mathscr{R}^\dagger(\alpha)\,\mathscr{P}_+(\theta)\mathscr{R}(\alpha) = \mathscr{P}_+(\theta+\alpha)\,, \tag{30}$$

and, in particular, using also (26),

$$\mathscr{R}^\dagger\left(-\frac{\pi}{2}\right)\mathscr{P}_+(0)\mathscr{R}\left(-\frac{\pi}{2}\right) = \mathscr{P}_+\left(-\frac{\pi}{2}\right) = \mathscr{P}_-(0)\,. \tag{30a}$$

7. – Cascaded systems.

The result of cascading the various devices discussed thus far is of course simply obtained by the consecutive application of the respective operators. One special case of such a cascaded system of particular interest is the compensator followed by a polarizer. Computing the intensity at a point in the beam emerging from such an arrangement we obtain

$$\begin{aligned} I_K(\theta,\delta) = \mathrm{Sp}[\mathscr{P}_+(\theta)\mathscr{C}(\delta)\mathscr{J}\mathscr{C}^\dagger(\delta)\,\mathscr{P}_+(\theta)] = \mathrm{Sp}[\mathscr{P}_+^2(\theta)\mathscr{C}(\delta)\mathscr{J}\mathscr{C}^\dagger(\delta)] = \\ = \mathrm{Sp}[\mathscr{C}^\dagger(\delta)\,\mathscr{P}_+(\theta)\mathscr{C}(\delta)\mathscr{J}] = \mathrm{Sp}[\mathscr{K}(\theta,\delta)\mathscr{J}]\,, \end{aligned} \tag{31}$$

where we have set

$$\mathscr{K} = \mathscr{C}^\dagger\mathscr{P}_+\mathscr{C} = \mathscr{C}^{-1}\mathscr{P}_+\mathscr{C}\,. \tag{32}$$

The matrix $\mathscr{K}$ which, incidentally, arises from $\mathscr{P}_-$ through a rotation in Fourier space, is useful enough to be written out in full here for later reference. We find

$$\mathscr{K}(\theta,\delta) = \begin{pmatrix} \cos^2\theta & \exp\left[\frac{i}{2}\delta\right]\sin\theta\cos\theta \\ \exp\left[-\frac{i}{2}\delta\right]\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{33}$$

Note that (31) is also of the form (1), and that the effect of the combination is represented by an operation directly on $\mathscr{J}$. If (31) is written in detail, we obtain the familiar result for such an arrangement, as discussed by WOLF [2]:

$$I_K(\theta,\delta) = J_{xx}\cos^2\theta + J_{yy}\sin^2\theta + (J_{xy}\exp[-i\delta] + J_{yx}\exp[i\delta])\sin\theta\cos\theta\,. \tag{34}$$

Here J_{xx} etc., are the components of the coherency matrix.

8. – The state of the field.

The state of polarization of an electromagnetic field has been described is several ways: in terms of Stokes parameters, the degree of coherence between the x- and y-components of the field, and the degree of polarization. We shall discuss each of these modes of description in terms of the coherency matrix. Most of the relationships discussed in this section were pointed out by WOLF [2]. They are, however, obtained here from a quite different point of view which is being stressed as physically more meaningful in this discussion.

The customary set of Stokes parameters has been recently the subject of considerable discussion and it was pointed out by WOLF [2] that as usually defined they are not unique. This ambiguity was, however, removed by the introduction of the analytic signal representation. It was also shown recently by ROMAN [9] that the concept of Stokes parameters can be extended to the case of non-planar waves.

While it is the opinion of the authors that the elements of $\mathscr{J}$ are physically a more meaningful set of parameters, we will digress briefly and discuss the relation between $\mathscr{J}$ and the Stokes parameters.

Anticipating the discussion in Section **10**, we note that the coherency matrix is formally identical to the density matrix, and therefore the expansion of the density matrix given by FANO [10] may be used, *i.e.* we may set

$$\mathscr{J} = \tfrac{1}{2} \sum_{k=0}^{3} s_k \sigma_k \,, \tag{35}$$

where the σ_k are the Pauli matrices (10) and the s_k are the Stokes parameters. In keeping with the theme of this paper, we wish to express the Stokes parameters as derivable from $\mathscr{J}$, since they are observables of the field. Multiplying (35) by σ_i and taking the trace we obtain, using (9), the solution

$$s_i = \mathrm{Sp}[\sigma_i \mathscr{J}] \,, \qquad (i = 0, 1, 2, 3) \,. \tag{36}$$

Note that the Stokes parameters, as all measurables of the field, are obtained from an expression of the form (1).

Another parameter often used in the description of partially polarized fields is the degree of polarization, which may be defined as the ratio of the intensity of the completely polarized part of the radiation to the total intensity. It was shown by WOLF [2] that an unambiguous meaning can be given to the

[9] P. ROMAN: *Nuovo Cimento*, **13**, 974 (1959).
[10] U. FANO: *Phys. Rev.*, **93**, 121 (1954).

terms « completely polarized part » and « unpolarized part » and a unique expression for the degree of polarization can be obtained in terms of the invariants of $\mathscr{J}$.

The situation is analogous to that which existed in the closely related theory of partial coherence. Here the ratio of the intensity of the « coherent part » of the radiation to the total intensity was proposed as a definition of the degree of coherence. However, a much clearer understanding of the physical situation was obtained by defining the degree of coherence in terms of a cross-correlation function (cfr. WOLF [11]). Following this lead from scalar coherence theory we shall propose here a new definition of the degree of polarization which will be seen to have the advantage that it involves only *directly* observable quantities.

Our definition is suggested by the following considerations. A strictly monochromatic field is completely polarized and completely coherent. Further, as pointed out by WOLF [1,2], the intensity formula (34) is formally identical with the expression for the intensity resulting from the superposition of two partially coherent beams. Now, the significant entity for describing interference phenomena involving partially coherent radiation is the normalised cross-correlation of the interfering disturbances, *i.e.* the degree of coherence. Thus our considerations suggest that *a significant quantity for the specification of the state of a partially polarized field is the degree of coherence between the x- and y- components of the field.* This quantity μ_{xy} may then be defined as the normalized cross-correlation of the x- and y-components of the field.

The x-component of the field is of course given by $\mathscr{P}_+(0)\mathscr{E}$, and the y-component *which is rotated as to interfere with the x-component* can be expressed as $\mathscr{P}(0)\mathscr{R}(-\pi/2)\mathscr{E}$. Since, as a natural extension of the intensity formula (5), we can express the cross-correlation of two fields as

$$K = \mathrm{Sp}\,\langle \mathscr{E}_1 \times \mathscr{E}_2^{\dagger} \rangle ,$$

the normalized cross-correlation between the aforementioned two field components may be written in the form

$$(37) \qquad \mu_{xy} = \frac{\mathrm{Sp}\,\langle [\mathscr{P}_+(0)\mathscr{E}] \times [\mathscr{P}_+(0)\mathscr{R}(-\pi/2)\mathscr{E}]^{\dagger} \rangle}{\{\mathrm{Sp}\,\langle [\mathscr{P}_+(0)\mathscr{E}] \times [\mathscr{P}_+(0)\mathscr{E}]^{\dagger} \rangle \cdot \mathrm{Sp}\,\langle [\mathscr{P}_+(0)\mathscr{R}(-\pi/2)\mathscr{E}] \times [\mathscr{P}_+(0)\mathscr{R}(-\pi/2)\mathscr{E}]^{\dagger} \rangle\}^{\frac{1}{2}}} .$$

Here the normalizing denominator represents the square root of the product of the intensities associated with the two interfering fields.

(11) E. WOLF: *Proc. Roy. Soc.*, A **230**, 96 (1955).

Permuting cyclically the arguments of the traces, utilizing the properties (23), (24) and (30a) of the projection operators, the unitarity of $\mathscr{R}$ and the Hermiticity of $\mathscr{P}$, and applying the « associative law » that was used first in the derivation of (14), this expression can be considerably simplified.

We obtain, in view of (6)

$$\mu_{xy} = \frac{\mathrm{Sp}\,[\mathscr{R}(\pi/2)\,\mathscr{P}_+(0)\mathscr{J}]}{\{\mathrm{Sp}\,[\mathscr{P}_+(0)\mathscr{J}]\cdot\mathrm{Sp}\,[\mathscr{P}_-(0)\mathscr{J}]\}^{\frac{1}{2}}}\,. \tag{38}$$

By the application of the Schwarz inequality it may be shown that the nodulus of μ_{xy} is bounded by zero and one and these extreme values are characteristic of incoherence and coherence respectively.

Since, apart from $\mathscr{I}$, the matrices in (38) are real, we may write

$$|\mu_{xy}|^2 = \frac{\mathrm{Sp}\,[\mathscr{R}(\pi/2)\,\mathscr{P}_+(0)\mathscr{J}]\cdot\mathrm{Sp}\,[\mathscr{R}(\pi/2)\,\mathscr{P}_+(0)\,\mathscr{I}^*]}{\mathrm{Sp}\,[\mathscr{P}_+(0)\mathscr{J}]\cdot\mathrm{Sp}\,[\mathscr{P}_-(0)\mathscr{I}]}\,. \tag{39}$$

Our expression for μ_{xy} is somewhat arbitrary, because for a different selection of the x- and y-directions we would of course obtain a different value. Clearly μ_{xy} is not yet a satisfactory measure for the state of polarization. However, we shall now show that there always exists a coordinate frame for which a maximum value of $|\mu_{xy}|^2$ is obtained.

The value of $|\mu_{xy}|^2$ referred to a coordinate system (XY) making an angle θ with the original one may be obtained from (39) by simply noting that in the new system the projection operator $\mathscr{P}(0)$ must be replaced by $\mathscr{P}(\theta)$. Thus

$$|\mu_{xy}(\theta)|^2 = \frac{\mathrm{Sp}\,[\mathscr{R}(\pi/2)\,\mathscr{P}_+(\theta)\mathscr{J}]\cdot\mathrm{Sp}\,[\mathscr{R}(\pi/2)\,\mathscr{P}_+(\theta)\mathscr{I}^*]}{\mathrm{Sp}\,[\mathscr{P}_+(\theta)\mathscr{J}]\cdot\mathrm{Sp}\,[\mathscr{P}_-(\theta)\mathscr{J}]}\,. \tag{40}$$

The straightforward but rather lengthy maximization of (40) with respect to θ yields the following condition on θ:

$$\mathrm{tg}\,2\theta_m = \frac{J_{yy} - J_{xx}}{J_{xy} + J_{yx}}, \tag{41}$$

where J_{xy} etc. are the elements of the coherency matrix with respect to the original choice of axes. Since $\mathscr{J}$ is Hermitian, it follows from (41) that there always exists a *real* θ such $|\mu_{xy}|$ is maximal.

Before proceeding we point out that the angle θ_m which maximises $|\mu_{xy}|$ is also the angle through which the coordinates must be rotated in order that the intensities associated with the x- and y-components are equal. This is

easily shown by simply demanding that

$$\mathrm{Sp}\,[\mathscr{P}_+(\theta)\mathscr{I}] = \mathrm{Sp}\,[\mathscr{P}_-(\theta)\mathscr{I}]\,,$$

and solving for θ.

We are now in a position to define the degree of polarization of the beam. *The degree of polarization is defined as the maximum value of the modulus of the degree of coherence*, $|\mu_{xy}|$, maximized with respect to θ. According to this definition we obtain the degree of polarization, P, by substituting the value of θ_m given by (41) into (40). We find after some calculation that

$$P = \sqrt{1 - \frac{4\det\mathscr{I}}{(\mathrm{Sp}\,\mathscr{I})^2}}\,. \tag{42}$$

Since $\det\mathscr{I}$ and $\mathrm{Sp}\,\mathscr{I}$ are invariants, it is clear that P as given by (42) is independent of the choice of axes.

We note that equ. (42) is the result derived by WOLF [2] starting from the definition

$$P = \frac{I_{\mathrm{pol}}}{I_{\mathrm{tot}}}\,. \tag{43}$$

Since it will be of help in deriving an experimental scheme for measuring $\mathscr{I}$ (see Section 9), we will digress now briefly to consider the determination of P from the definition (43)

Since $\mathscr{I}$ is Hermitian, it is possible to diagonalize it with a unitary matrix $\mathscr{D}$, *i.e.* there exists a $\mathscr{D}$ such that $\mathscr{D}^\dagger = \mathscr{D}^{-1}$ and

$$\mathscr{D}\mathscr{I}\mathscr{D}^{-1} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \tag{44}$$

where the eiginvalues are given by

$$\lambda_\pm = \frac{1}{2}\,\mathrm{Sp}\,\mathscr{I}\left\{1 \pm \sqrt{1 - \frac{4\det\mathscr{I}}{(\mathrm{Sp}\,\mathscr{I})^2}}\right\}. \tag{45}$$

Physically (44) implies that there exists a frame of reference (coordinate system and relative phase difference) such that the cross-correlation terms vanish. Thus the operator $\mathscr{D}$ must be of the form

$$\mathscr{D}(\alpha, \delta) = \mathscr{R}(\alpha)\mathscr{C}(\delta)\,. \tag{46}$$

The appropriate diagonalizing angles α and δ are readily found to be given by

$$\exp[2i\delta] = J_{yx}/J_{xy}\,, \tag{47a}$$

and

$$\operatorname{tg} 2\alpha = \frac{1}{\cos\delta}\frac{J_{xy}+J_{yx}}{J_{xx}-J_{yy}} = \frac{J_{xy}+J_{yx}}{J_{xx}-J_{yy}}\sqrt{\frac{2}{1+\operatorname{Re}(J_{yx}/J_{xy})}}\,. \tag{47b}$$

Proceeding, we may now write

$$\begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix} = \lambda_-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + (\lambda_+ - \lambda_-)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{48}$$

The first term on the right of (48) represents evidently a completely unpolarized beam and the second a completely polarized beam. Furthermore, this decomposition is unique. Substituting from (48) into the definition (43) we obtain

$$P = \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-}\,; \tag{49}$$

and substituting from (45) into (49) we obtain the same expression (42) for P as derived above from our definition. Hence the equivalence of the two definitions is established.

9. – Measurement of $\mathscr{J}$.

In order to give a simple and direct physical interpretation to each of the four measurements involved in the determination of $\mathscr{J}$, we consider first a decomposition of $\mathscr{J}$.

Setting

$$J_{xy} = \beta + i\gamma\,, \qquad \text{hence} \qquad J_{yx} = \beta - i\gamma\,,$$

we may separate $\mathscr{J}$ into a real and imaginary part as

$$\mathscr{J} = \begin{pmatrix}J_{xx} & \beta\\ \beta & J_{yy}\end{pmatrix} + i\gamma\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \equiv \mathscr{J}^r + \gamma\sigma_3\,.$$

We note that

$$\operatorname{Sp}[\mathscr{P}_+(\theta)\mathscr{R}(\alpha)\sigma_3\mathscr{R}^\dagger(\alpha)] = 0$$

for *all* θ and α. In view of the discussion in the previous sections this implies that *the imaginary part of* $\mathscr{J}$ *does not contribute to the intensity unless a compensator is involved in the device through which the radiation has passed.* We may therefore describe the interaction of a partially polarized beam with a polarizer and a rotator solely in terms of the real (and symmetric) matrix $\mathscr{J}^r$.

Thus the intensity in a beam passed by a polarizer is given by

$$I_p(\theta) = \mathrm{Sp}\,[\mathscr{P}_+(\theta)\mathscr{J}^r]\,. \tag{51}$$

Since $\mathscr{J}^r$ is symmetric, it follows from (47*a*) and (46) that the matrix $\mathscr{S}$ which diagonalizes $\mathscr{J}^r$ is a pure rotation operator $\mathscr{R}$. Hence, denoting the eigenvalues of $\mathscr{J}^r$ by $\lambda^r_\pm$, we may write

$$\mathscr{R}(\alpha_d)\mathscr{J}^r\mathscr{R}^{-1}(\alpha_d) = \begin{pmatrix} \lambda^r_+ & 0 \\ 0 & \lambda^r_- \end{pmatrix}, \tag{52}$$

where the diagonalizing angle α_d is, according to (47*b*), determined by the condition

$$\mathrm{tg}\,2\alpha_d = \frac{2\beta}{J_{xx} - J_{yy}}\,. \tag{53}$$

Using (52) and cyclically permuting the argument of the trace, (51) may be written as

$$I_p(\theta) = \mathrm{Sp}\left\{\mathscr{R}(\alpha_d)\mathscr{P}_+(\theta)\mathscr{R}^+(\alpha_d)\begin{pmatrix} \lambda^r_+ & 0 \\ 0 & \lambda^r_- \end{pmatrix}\right\} = \mathrm{Sp}\left\{\mathscr{P}_+(\theta - \alpha_d)\begin{pmatrix} \lambda^r_+ & 0 \\ 0 & \lambda^r_- \end{pmatrix}\right\}, \tag{54}$$

where use was made of (30). Taking now in particular $\theta = \alpha_d$, we obtain

$$I_p(\alpha_d) = \lambda^r_+\,, \tag{55a}$$

and similarly, for $\theta = \alpha_d + \pi/2$ we get

$$I_p\left(\alpha_d + \frac{\pi}{2}\right) = \lambda^r_-\,. \tag{55b}$$

We now notice that *the same angle* α_d *which diagonalizes* $\mathscr{J}^r$ *makes also the intensity* $I_p(\theta)$ *an extremum.* This can be shown simply by differentiating (51) with respect to θ. We obtain then the condition

$$\mathrm{tg}\,2\theta_e = \frac{2\beta}{J_{xx} - J_{yy}}\,, \tag{56}$$

which is indeed identical with (53). Whether this extremum is a maximum or a minimum, depends on the sign of β. Combining this result with (55a, b) we can set

$$\begin{cases} \lambda_+^r = I_{e_1}, \\ \lambda_-^r = I_{e_2}; \end{cases} \tag{57}$$

where I_{e_1} and I_{e_2} denote the first and second extremal intensities (a maximum and a minimum if $\beta > 0$, and the opposite way round if $\beta < 0$). These occur at an angle $\theta_{e_1} = \alpha_d$ and $\theta_{e_2} = \alpha_d + \pi/2$ respectively, where α_d is given by (53).

The physical significance of the above mathematical considerations is obvious. To measure the elements J_{xx}, J_{yy}, and $\beta = \mathrm{Re}\, J_{xy}$ of $\mathscr{J}$, we use a polarizer and find first the smallest angle α_d for which we obtain an extremal intensity I_{e_1}. We measure this intensity and then turn the polarizer to the setting $\alpha_d - \pi/2$ and measure the new extremal intensity I_{e_2}. By (57) the two intensities give us directly the two eigen-values λ_-^r; then, knowing also the angle α_d we compute

$$\mathscr{J}^r = \begin{pmatrix} J_{xx} & \beta \\ \beta & J_{yy} \end{pmatrix} = \mathscr{R}^{-1}(\alpha_d) \begin{pmatrix} I_{e_1} & 0 \\ 0 & I_{e_2} \end{pmatrix} \mathscr{R}(\alpha_d),$$

where $\mathscr{R}$ is given by (20).

If we like, we can now compute the first three Stokes parameters. According to (36) and (10)

$$\begin{cases} s_0 = J_{xx} + J_{yy}, \\ s_1 = J_{xx} - J_{yy}, \\ s_2 = 2\beta. \end{cases} \tag{58}$$

It follows that if one is interested only in experiments which do not involve compensators, then these three parameters specify the beam completely.

For a complete description we also require, however, the fourth parameter $\gamma' = \mathrm{Im}\, J_{xy}$. This can be obtained by an additional measurement involving a compensator and a polarizer. The intensity is then given by (31). Using a half-wave plate for which $\delta = \pi$ and setting the polarizer at an angle $\theta = \pi/4$ with respect to the x-axis, we obtain, using (33) and the notation of equ. (50)

$$I_K = \tfrac{1}{2}(J_{xx} + J_{yy}) + \gamma';$$

hence

$$\gamma = I_K - \tfrac{1}{2} I_{\mathrm{inc}}, \tag{59}$$

where $I_{\text{inc}} \equiv J_{xx} + J_{yy} = \text{Sp}\,\mathscr{I}$ is the incident intensity. Since J_{xx} and J_{yy} have been measured already in the previous experiment, the sole determination of the intensity I_K suffices to determine γ. As a matter of fact, settings other than $\theta = \pi/4$, $\delta = \pi$ could serve our purpose just as well. Knowing γ, the fourth Stokes parameter can be expressed as

$$s_3 = 2\gamma \,. \tag{60}$$

10. – Concluding remarks.

In the above discussion we have seen that a suitable and comprehensive description of the properties and the interactions of statistical radiation can be obtained in terms of the coherency matrix $\mathscr{I}$. It is interesting to note that both the definition and the properties of $\mathscr{I}$ resemble very much those of the statistical density matrix ϱ introduced for the characterization of statistical mixtures of quantum mechanical systems by VON NEUMANN (*). The application of the density matrix in the description of phenomena connected with electron- and photon-polarization has been discussed in great detail by TOLHOEK (12).

The main resemblance between the use of ϱ and of $\mathscr{I}$ is reflected in the fact that the expectation value of a physical observable is given in terms of the density matrix by

$$F = \langle \mathscr{F} \rangle = \text{Sp}\,[\mathscr{F}\varrho] \,,$$

while the observable value of a physical characteristic connected with a measurement is given in the present formalism by the analagous expression (1). In particular, as one finds easily by combining (15), (18), (21), (27), the « operator of the intensity » is given, in view of (5) and (1), by the operator

$$\mathscr{I} = \mathscr{A}^2(\eta)\,\mathscr{K}(\theta + \alpha, \delta) \,, \tag{61a}$$

where use was made of (30) and the notation (32). Similarly, the « operator of the Stokes parameter s_i » is, according to (36),

$$\mathscr{S}_i = \sigma_i \,. \tag{61b}$$

(*) For a detailed account see, for example, R. C. TOLMAN: *The principles of statistical mechanics* (Oxford, 1938), p. 327.

(12) H. A. TOLHOEK: *Rev. Mod. Phys.*, **28**, 277 (1956).

Furthermore, specifying the statistical ensemble in a new representation of variables, *i.e.*, physically, performing a certain measurement on the system by letting it interact with some device, amounts to a similarity transformation on ϱ:

$$\varrho \to \varrho' = \mathcal{O}\varrho\mathcal{O}^{-1} = \mathcal{O}\varrho\mathcal{O}^{\dagger} . \tag{62a}$$

On the other hand, we have seen in Sect. 3 and 6 that the interaction of the statistical beam amounts to a transformation

$$\mathcal{I} \to \mathcal{I}' = \mathcal{O}\mathcal{I}\mathcal{O}^{\dagger} . \tag{62b}$$

There are, however, some important differences between the two formalisms. For one thing, ϱ is defined in terms of ensemble-averages, while the definition of $\mathcal{I}$ involves time-averages. Even though it is true that — by virtue of the ergodic hypothesis — the two averaging procedures are normally equivalent, this equivalence will break down in the limiting case of a strictly monochromatic beam.

Another difference is that the transforming operator in (62*a*) is always unitary, but in (62*b*) not necessarily so. For example, the operator $\mathcal{P}_{\perp}$ characterizing a polarizer is not. This circumstance arises from the fact that in our discussions we have confined our attention to describing the emergent beam only, while the complete specification of the interaction should involve also the non-transmitted (reflected or absorbed) component of the radiation as well.

RIASSUNTO (*)

La matrice di coerenza di un'onda piana quasi-monocromatica viene dedotta da una rappresentazione matriciale del segnale analitico associato al campo elettrico. Si mostra che se la radiazione passa attraverso un apparecchio fisico, quale un compensatore, un assorbitore, un rotatore o un polarizzatore, l'effetto di questa interazione può essere completamente descritto in termini di operatori opportunamente scelti che trasformano direttamente la matrice di coerenza. Il grado complesso di coerenza viene definito in termini degli operatori suddetti, e da questo viene dedotta un'espressione che caratterizza il grado di polarizzazione. Si mostra che la quantità così dedotta è uguale a quella ottenuta con la definizione più usuale. Si descrive un esperimento che può servire a misurare le componenti della matrice di correlazione.

(*) *Traduzione a cura della Redazione.*

Formulation of the Causality Requirement (*).

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(ricevuto il 19 Ottobre 1959)

Summary. — A recently published formulation of the causality requirement is considered. This formulation is shown to differ in important respects from the usual requirement of « no output before input » and certain difficulties of physical interpretation are discussed. The formulation is shown to be more restrictive than the usual strict causality and to exclude bound states and certain types of resonances that occur in theories of physical interest. The discussion of the Klein-Gordon wave in terms of the characteristic momentum variable is analyzed further and the way in which the formulation leads to a physically unacceptable extension of the domain of analyticity is described.

A recent paper [1] has presented a new approach to the formulation of the causality requirement in scattering problems. The purpose of this note is to exhibit important differences between the formulation of reference [1] and the usual meaning of the causality principle.

The causality condition of reference [1] is illustrated by applying it to the problem of the scattering of a Klein-Gordon wave in the space of one-time dimension and one-space dimension. This is a convenient problem for the study of the meaning of the causality requirement, for many of the mathematical complications of the full quantum field theoretic treatment are avoided. It is possible in this simple problem to select a physically meaningful formu-

(*) This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command.

(**) On leave of absence from Christ's College, Cambridge, U. K.

(1) I. E. Segal: *Phys. Rev.*, **109**, 2191 (1958).

lation of causality that is appropriate for later extension to a complete quantum field theory. We use units where $\hbar = c = 1$. The general solution of this Klein-Gordon equation $(\partial^2\varphi/\partial x^2 - \partial^2\varphi/\partial t^2 - m^2\varphi)$ is obtained by a superposition of the plane waves $\exp[ik_1x - ik_0t]$, where $k_0^2 - k_1^2 = m^2$. In reference [1] it is shown that these plane waves can be conveniently expressed in terms of a new variable called the characteristic momentum $s = \frac{1}{2}(k_0 + k_1)$, in terms of which the Fourier component can be written as $\exp[is(x-t) - is_1(x+t)]$, where $s_1 = m^2/4s = \frac{1}{2}(k_0 - k_1)$. The general superposition of oscillatory waves is thus given by:

$$\varphi(x, t) = \int_{-\infty}^{+\infty} f(s) \exp[is(x-t) - is_1(x+t)] \cdot ds/|s|, \tag{1}$$

where $f(s)/|s|$ is an arbitrary single-valued function of the real characteristic momentum.

In reference [1] a scattering problem is formulated by considering an input wave $\varphi_{in}(x, t)$ to give rise to an output wave $\varphi_{out}(x, t)$, both expressed in the form of equation (1), such that «for forward scattering» the transforms $f_{in}(s)$ and $f_{out}(s)$ are related by

$$f_{out}(s) = (1 + \zeta(s))\, f_{in}(s) .$$

Here $\zeta(s)$ measures the scattering amplitude. The usual methods of complex variable theories are then applied by considering inputs such that $f_{in}(s)/|s|$ is the boundary value of an analytic and bounded function in the upper half of the complex s-plane.

We point out first that the condition that $f_{in}(s)/|s|$ extend into an analytic function forces the input to contain waves which move in both positive and negative senses along the x-axis. The individual Fourier components for real s move in the positive x direction when $k_0k_1 > 0$ or $s^2 > m^2/4$, while the waves moving in the negative x direction are given by $k_0k_1 < 0$ or $s^2 < m^2/4$. Thus a wave of form (1) containing only components moving in the positive x direction must be such that $f(s) = 0$ for $f^2 < m^2/4$. Such a function can extend analytically into the upper half s-plane only if it is identically zero. Hence the inputs considered in reference [1] contain waves moving in both senses. This fact is the principal source of the essential difference between reference [1] and the usual analysis of causality.

The causality condition used in reference [1] is:

(3a) if $\varphi_{in}(x, t) = 0$ for $x = -t > 0$,

(3b) then $\varphi_{out}(x, t) = 0$ for $x = -t > 0$.

In other words, it is assumed that, if the input vanishes along the backward light ray $x = -t$, $x > 0$, then the output must vanish along this ray.

We now compare the condition (3) with usual causality condition which is expressed by the physical assumption that there can be « no output before the input »; that is, if $\varphi_{in}(x, t)$ vanishes for *all* events x, t in the past of the space-time event 0, then $\varphi_{out}(x, t)$ must vanish for all events in the past of 0:

Thus:

(4a) if $\varphi_{in}(x, t) = 0$ for all $|x| < -t$, $t < 0$,

(4b) then $\varphi_{out}(x, t) = 0$ for all $|x| < -t$,

Condition (3) can be shown to be much more restrictive than the usual causality condition (4). First we note that there are many input functions $\varphi_{in}(x, t)$ which satisfy (3a) but do not fulfil (4a). It is easy to construct functions which vanish along a given ray but which are non-zero at most points in the interior of the past light cone, for example for $m = 0$, the function $\varphi_{in}(x, t) = [(t+x)/(t-x)]^{n/2} J_n(m\sqrt{t^2 - x^2})$ for any positive integer n satisfies (3a) but not (4a); for $m = 0$, $\varphi_{in}(x, t) = \psi(x+t)$ for any function ψ with $\psi(0) = 0$ satisfies (3a). Thus an input can satisfy (3a) and still have non-vanishing values at events located densely in all space-time regions; the usual formulation of causality would permit such an input to generate output waves at any point in space and time. Hence the condition (3) places a physical restriction on the scatterer in addition to the normal concept of causality. In the following we will analyse the limitations of the usual causality principle and show how much stronger condition (3) is; in particular, we will exhibit how condition (3) is too restrictive to be physically acceptable in that it would exclude bound states and also some types of resonances which occur in many causal theories of physical interest.

In order to be able to discuss the concept of « forward scattering » in a physically meaningful way, we consider an input wave composed only of waves moving in one sense, *e.g.*, along the positive x-axis. To analyse the usual physical concept of strict causality (« no output before input »), we must be able to construct an input signal which vanishes at a given reference point for all times before some initial time [2]. However, a non-trivial input wave packet which moves only in one sense and also vanishes for a time interval can be formed from solutions of the Klein-Gordon equation only if we include Fourier components of *all* frequency bands, including the solutions for $k_0' < m$. Thus the input wave is expressed as

$$\varphi_{in}(x, t) = \int_{-\infty}^{+\infty} F_{in}(k_0) \exp[i k_1 x - i k_0 t] \, dk_0 , \tag{5}$$

[2] See, for example, J. S. Toll: *Phys. Rev.*, **104**, 1760 (1956).

where $k_1 = +\sqrt{k_0^2 - m^2}$ for $k_0 > m$, and is extended analytically through the upper half of the complex k_0-plane to give $k_1 = i\sqrt{m^2 - k_0^2}$ for real k_0 with $|k_0| < m$ and $k_1 = -\sqrt{k_0^2 - m^2}$ for real $k_0 < -m$. The oscillatory components (real k_0, $|k_0| > m$) represent waves moving freely in the positive x direction, since $k_0 k_1 > 0$ for all these components. The components for real k_0, with $|k_0| < m$, decrease with increasing x as $\exp[-\sqrt{m^2 - k_0^2}\,x]$; these components can be interpreted as the wave functions for bound states of a source located on the distant negative x axis ([3]). For purposes of comparison with reference ([1]), the integration over k_0 in (5) can be transformed to an integral over s. Then equation (5) becomes of the form of equation (1), except that the contour of integration becomes the contour C in the complex s-plane, as indicated in Fig. 1. The forward scattered outgoing wave can also be defined by an integral

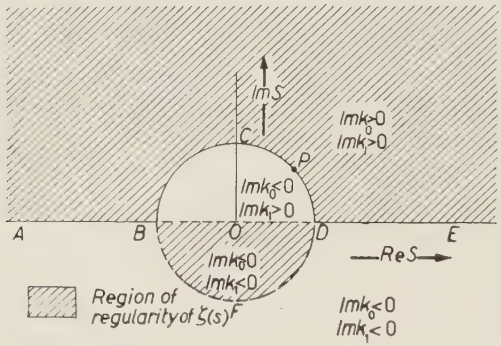

Fig. 1. – *Complex characteristic momentum plane*: SEGAL has shown that it is convenient to introduce the « characteristic momentum » s in terms of which the usual momentum k_1 and energy k_0 are given by $k_1 = s - m^2/4s$ and $k_0 = s + m^2/4s$, where $k_0^2 = k_1^2 + m^2$, $s = (k_0 + k_1)/2$. The full k_0-plane is mapped twice into the s-plane, once into the upper half s-plane, where Im $k_1 > 0$ and once into the lower half s-plane, where Im $k_1 < 0$. Thus the double-valuedness of k_1 in terms of k_0 (or vice-versa) is removed as both k_0 and k_1 become single-valued functions of s. The KLEIN-GORDON waves $\exp[ik_1 x - ik_0 t]$ are thus uniquely specified by the value of s. Real values of s correspond to oscillatory waves ($s^2 > m^2/4$ indicates positive x direction, while $s^2 < m^2/4$ indicates negative x direction). The circular perimeter $|s| = m/2$ corresponds to real frequencies such that $|k_0| < m$; those on the upper semicircle BCD represent waves decreasing exponentially with positive x, while points of BFD give exponential decrease toward negative x. Thus all points of the contour C, following the path $ABCDE$, represent waves oriented in the positive x direction. It is shown in the text that causality is satisfied for such waves if and only if the scattering amplitude $\zeta(s)$ is analytic and appropriately bounded for complex s above the countour C; singularities may occur on the perimeter, such as a bound state singularity at the point P. Causality for waves toward negative x implies correspondingly that $\zeta(s)$ is analytic and appropriately bounded inside the semicircle $BODF$.

([3]) For further discussion, see J. M. KNIGHT and J. S. TOLL: *Ann. Phys.*, 3, 49 (1958) or A. BOHR and B. MOTTLESON (to be published).

of the type of equation (5) and the relation between the ingoing and outgoing waves is given in terms of the forward scattering amplitude ζ by equation (2).

The concept of causality can now be applied in conventional fashion. Choose any input of form (5) such that $\varphi_{\text{in}}(x, t) = 0$ for all $x > t$ and then require that the corresponding $\varphi_{\text{out}}(x, t) = 0$ for all $x > t$. In the usual manner [2,3] it is then easily shown that the scattering amplitude ζ must extend into an analytic function in the upper half of the complex k_0-plane which is bounded in this half plane by its values on the real axis. If ζ is not square integrable on the real k_0-axis, appropriate subtractions of poles and polynomials should be made [1] to obtain a square integrable function ξ, and then ξ is shown to be a causal transform in k_0, *i.e.*, an analytic function in the upper half k_0-plane the square integral of which, along any line parallel and above the real axis, is bounded by the square integral along the real axis. Then the real and imaginary parts of ξ determine each other through Hilbert transform integrals, and a dispersion relation for the scattering amplitude ζ can thus be derived as a necessary and sufficient condition for the usual formulation of the causality principle [2]. The most general case, in which ξ is a tempered distribution on the real k_0-axis, can also be shown to give rise to a dispersion relation with appropriate subtractions [5].

Thus the usual causality condition is equivalent to the analyticity and appropriate boundedness of ζ in the upper half k_0-plane (on the sheet where $\operatorname{Im} k_1 \geq 0$), which in turn maps into the portion of the complex s-plane which is above the contour C. However, $\zeta(s)$ need not be analytic on or below the contour C; it may have singularities on the real s-axis or on the perimeter or in the interior of the circle $|s| \leq m/2$. Indeed bound states in the scatterer normally lead to poles of ζ at values of k_0 with $|k_0| < m$, and $k_1 = i\sqrt{m^2 - k_0^2}$, and these are mapped into the perimeter of the semicircle $\boldsymbol{S}$ (defined by $|s| \leq m/2$, $\operatorname{Im} s > 0$).

In reference [1] it is shown that condition (3) is satisfied if and only if $\zeta(s)$ is analytic and appropriately bounded in the whole upper half s-plane. Thus condition (3) is more resctrictive than the normal causality condition in that it adds the perimeter and interior of the semi-circle $\boldsymbol{S}$ to the region of regularity. In particular, it therefore excludes bound states or other singularities within $\boldsymbol{S}$, and this explains why the dispersion relations given in reference [1] do not contain the conventional bound state or coupling constant terms.

It is easy to construct examples of scattering amplitudes satisfying the usual causality and unitarity requirements and containing no bound state terms

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[4] See, for example, Ref. [2], Sect. 5.

[5] J. G. Taylor: *Annals of Physics*, 5, 391 (1958).

but still violating condition (3). For example, let:

$$\zeta(s) = \frac{k_0 - im}{k_0 + im} - 1 = \frac{-2ims}{\left(s + [(\sqrt{2}+1)/2]\,im\right)\left(s - [(\sqrt{2}-1)/2]\,im\right)}. \tag{6}$$

This amplitude does not satisfy condition (3), for it has a pole at $s = \mathrm{Im}(\sqrt{2}-1)/2$ in the interior of the semicircle S. However, it is a physically reasonable scattering amplitude, for it only transforms an input such that $\varphi_{\mathrm{in}}(0, t) = \delta(t)$ into an output for which $\varphi_{\mathrm{out}}(0, t) = \delta(t) - 2m\,\theta(t)\exp[-mt]$ where $\theta(t) = 0$ for $t < 0$ and 1 for $t > 0$. In other words, this scatterer is characterized by the fact that inputs excite a transient which decays exponentially as $\exp[-mt]$; the response involves only positive time delays so that the usual conception of causality is satisfied.

We will now make comments on the extension of the above causality discussion to three-space dimensions.

However, here again we encounter the same difficulty of interpretation as in the one-dimensional case; the wave cannot vanish along the backward ray unless it contains components moving in both directions, in which case the concept of « forward scattering » is not well-defined. Thus there does not appear any useful or reasonable statement of this form of the causality requirement in three dimensions.

The analyticity properties of the S-matrix or the scattering amplitudes which follow from the usual from of « strict causality » in the case of three-space dimensions have been treated elsewhere ([3]). Such a discussion can be written in terms of the characteristic momentum s, with results exactly similar to those which have been described for one-space dimension in Fig. 1.

RIASSUNTO (*)

Si esamina una formulazione recentemente pubblicata del requisito di causalità. Si mostra che questa formulazione differisce sotto molti importanti aspetti dal requisito usuale « nessuna uscita prima di un'entrata » e si discutono alcune difficoltà nell'interpretazione fisica. Si mostra che questa formulazione è più restrittiva dell'usuale causalità ristretta e che esclude gli stati legati ed alcune risonanze che hanno luogo in teorie di interesse fisico. Si esamina ulteriormente la discussione dell'onda di Klein-Gordon in termini della variabile dell'impulso caratteristico e si descrive il processo per cui questa formulazione porta ad una estensione del dominio di analiticità fisicamente inaccettabile.

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Contributions à l'étude de l'interaction gravitationnelle des corps matériels.

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(ricevuto il 20 Ottobre 1959)

Résumé. — On donne quelques résultats concernant la structure et les liaisons des fonctions de Lagrange qui ont été établies pour le cas d'interaction gravitationnelle des points et des corps matériels. Dans le cas de deux sphères rigides en rotation (pour lesquelles la distribution de la densité est donnée par la formule de Roche) nous avons établi la fonction I, laquelle s'applique dans le cas où la distance des centres des sphères est d'au moins un ordre plus grande que la grandeur des rayons des sphères. Nous introduisons la notion du moment gravitationnel tout en démontrant que pour l'interaction des moments gravitationnels sont valables des expressions analogues à celles pour l'interaction des moments magnétiques.

Introduction.

Dans cette étude l'auteur tente une synthèse de certains de ses résultats obtenus dans ces dernières années dans le domaine des interaction gravitationnelles faibles ([1-4]). Voici ces résultats:

a) l'interaction gravitationnelle d'un point matériel en mouvement et d'un point en repos, peut être caractérisée par un quadri-vecteur-potentiel;

([1]) Z. GÁBOS: *An. st. Univ.* « Al. I. Cuza », *Iasi*, **1**, 1-2, 191 (1955).
([2]) Z. GÁBOS: *An. Univ. Babes-Bolyai*, Cluj, **2**, 1-2, 59 (1957).
([3]) Z. GÁBOS: *Studii si cerc. st. Iasi*, **9**, 1, 63 (1958); **9**, 2, 189 (1958); **10**, 1, (1959).
([4]) Z. GÁBOS: *An. st. Univ.* « Al. I. Cuza », *Iasi, en cours d'apparition.*

b) la condition relativement aux coordonnées de la théorie de la relativité généralisée joue le même rôle que la condition de Lorentz de l'électrodynamique. Par conséquent il est justifié d'utiliser de façon exclusive les coordonnées harmoniques, d'autant plus que ceci présente l'avantage de conduire à des expressions plus simples;

c) la fonction de Lagrange que V. A. Fok a établie pour le cas des corps rigides en rotation, peut encore être obtenue à l'aide de la fonction établie par I. G. Fichtenholtz pour les systèmes de points matériels;

d) nous avons établi la fonction L pour l'interaction de deux corps sphériques en rotation, pour lesquels la distribution de la densité est donnée par la formule de Roche. Le résultat s'applique dans le cas où la distance des centres des sphères est d'au moins un ordre plus grande que la grandeur des rayons des sphères;

e) nous introduisons la notion du moment gravitationnel tout en démontrant que pour l'interaction des moments gravitationnels sont valables des expressions analogues à celles pour l'interaction des moments magnétiques.

C'est I. G. Fichtenholtz [5] qui a établi la fonction de Lagrange pour un système de points matériels, et V. A. Fok [6] l'a établie pour le cas des corps rigides en rotation, dans l'hypothèse que la distance des centres de la masse de corps d'une forme quelconque est plus grande que leurs dimensions linéaires. Pour établir la fonction L, jusqu'ici on s'est servi de deux méthodes:

1) d'une méthode directe, en utilisant les équations d'Einstein, et

2) d'une méthode indirecte, en utilisant les relations existant entre la fonction L et l'expression ds^2.

Sans doute, la première méthode est la plus importante. Toutefois, la deuxième méthode peut servir aussi à approfondir et à compléter les résultats obtenus grâce à la première méthode. Nous nous proposons de démontrer dans cette étude la justesse de cette argumentation.

En général, les interactions gravitationnelles sont faibles. Par conséquent il suffit d'appliquer une approximation du deuxième ordre. Dans le cas des interactions faibles cette approximation nous conduit aux résultats expérimentaux mêmes et nous fournit une meilleure approximation que la théorie restreinte, mais une approximation moins bonne que la théorie généralisée de la relativité. Il est donc évident que l'on peut obtenir une approximation du deuxième ordre soit par la généralisation des résultats dus à la theorie re-

[5] I. G. Fichtenholtz: *J.E.T.F. Moscou*, **20**, 232 (1950); **32**, 1098 (1957).
[6] V. A. Fok: *Teoria prostranstva, vremeni i tiagotenia*, (Moscou, 1955), p. 288-384.

streinte ([7-9,1,2]), soit par l'application des méthodes approximatives dans le cas des équations de la théorie généralisée de la relativité ([10-13]).

1. – Interaction gravitationnelle d'un point matériel en mouvement et d'un point en repos.

Nous utilisons la relation

$$L\,\mathrm{d}t = -\,m_0 c\,\mathrm{d}s\,, \tag{1}$$

où m_0 est la masse de repos du point en mouvement. Les expressions $\mathrm{d}s^2$ utilisées ont la forme générale:

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 - g_{11}\,\mathrm{d}r^2 - g_{22}\,\mathrm{d}\theta^2 - g_{33}\,\mathrm{d}\varphi^2 = \tag{2}$$

$$= c^2(1+\alpha)\,\mathrm{d}t^2 - (1+\beta)\,\mathrm{d}r^2 - r^2(1+\gamma)(\mathrm{d}\theta^2 + \sin^2\Theta\,\mathrm{d}\varphi^2),$$

où α, β, γ sont des fonctions de la distance des points matériels (r). Vu que nécessairement

$$\lim_{r\to\infty}\alpha(r) = \lim_{r\to\infty}\beta(r) = \lim_{r\to\infty}\gamma(r) = 0\,,$$

donc α, β, γ peuvent être développés en série suivant les puissances successives de λ/r. λ est le rayon gravitationnel du point matériel en repos

$$\lambda = \frac{KM_0}{c^2}\,.$$

Pour le cas d'une approximation du deuxième ordre

$$\alpha = \alpha_0\frac{\lambda}{r} + \alpha_1\frac{\lambda^2}{r^2}\,, \qquad \beta = \beta_0\frac{\lambda}{r}\,, \qquad \gamma = \gamma_0\frac{\lambda}{r}\,,$$

En tenant compte de la relation $v^2 = \dot r^2 + r^2\dot\Theta^2 + r^2\sin^2\Theta\cdot\dot\varphi^2$ où $\dot r = \mathrm{d}r/\mathrm{d}t$, (2) peut être remplacé par

$$\mathrm{d}s^2 = c^2\left(1 + \alpha_0\frac{\lambda}{r} + \alpha_1\frac{\lambda^2}{r^2} - \frac{v^2}{c^2} - \frac{\beta_0-\gamma_0}{c^2 r}\,\dot r^2 - \frac{\gamma_0}{c^2 r}\,v^2\right)\mathrm{d}t^2\,.$$

([7]) O. ONICESCU: *Rev. Univ.* « C. I. Parhon » *Bucaresti, seria st. nat.*, **3**, 31 (1953).

([8]) T. T. VESCAN: *Studii si cerc. de fizică*, **5**, 3-4, 249 (1954).

([9]) T. T. VESCAN, A. WEISZMANN et I. GOTTLIEB: *Lucr. consf. de geom. dif.* 9-12 Iunie 1955, *Timisoara*, 1956, p. 341.

([10]) A. EINSTEIN et J. GROMMER: *Sitzb. Berl. Akad.*, no. 1/4, 2 (1927).

([11]) A. EINSTEIN, L. INFELD et B. HOFFMANN: *Ann. Math.*, **39**, 65 (1938).

([12]) V. A. FOK: *J.E.T.F. Moscou*, **9**, 375 (1949).

([13]) L. INFELD: *Acta Phys. Polonica*, **13**, 187 (1954).

Les deuxième et quatrième termes de l'expression en parenthèse ont l'ordre de grandeur λ/r; les troisième, cinquième et sixième ont λ^2/r^2 pour ordre de grandeur. Lorsque l'interaction est faible $L = -m_0 c(\mathrm{d}s/\mathrm{d}t)$ en la remplaçant par un développement limité au deuxième ordre, nous obtenons

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - V,$$

où

$$V = \frac{m_0 \lambda c^2}{2r}\left[\alpha_0 + \left(\alpha_1 - \frac{\alpha_0^2}{4}\right)\frac{\lambda}{r} + \left(\frac{\alpha_0}{2} - \gamma_0\right)\frac{v^2}{c^2} - (\beta_0 - \gamma_0)\frac{\dot{r}^2}{c^2}\right].$$

Pour les coordonnées ordinaires et harmoniques nous avons respectivement:

$$\alpha_0 = -2\,, \qquad \alpha_1 = 0\,, \qquad \beta_0 = 2\,, \qquad \gamma_0 = 0\,,$$

et

$$\alpha_0 = -2\,, \qquad \alpha_1 = 2\,, \qquad \beta_0 = \gamma_0 = 2\,.$$

En utilisant l'expression de la fonction L, nous écrivons, pour l'énergie et pour le vecteur généralisé de la quantité de mouvement

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} + \varphi\,, \tag{4}$$

où φ est l'énergie du champ gravitationnel:

$$\varphi = V - \dot{x}_i \frac{\partial V}{\partial \dot{x}_i} = \frac{m_0 \lambda c^2}{2r}\left[\alpha_0 + \left(\alpha_1 - \frac{\alpha_0^2}{4}\right)\frac{\lambda}{r} - \left(\frac{\alpha_0}{2} - \gamma_0\right)\frac{v^2}{c^2} + (\beta_0 - \gamma_0)\frac{\dot{r}^2}{c^2}\right],$$

et

$$P_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{m_0 v_i}{\sqrt{1 - v^2/c^2}} + A_i\,, \tag{5}$$

avec

$$A_i = \left(\gamma_0 - \frac{\alpha_0}{2}\right)\frac{m_0 \lambda v_i}{r} + (\beta_0 - \gamma_0)\frac{m_0 \lambda \dot{r}}{r^2}\, x_i\,.$$

La fonction L peut être donnée, de la même façon que dans la théorie de la relativité restreinte, au moyen des fonctions A_i, φ:

$$L = -m_0 \sqrt{1 - \frac{v^2}{c^2}} + A_i v_i - \varphi\,.$$

Les fonctions A_1, A_2, A_3, $A_4 = (i/c)\varphi$ sont les composants d'un quadrivecteur potentiel, pourvu qu'elles satisfassent à deux conditions:

α) l'expression

$$A_i \, \mathrm{d}x_i - \varphi \, \mathrm{d}t$$

est un invariant de Lorentz;

β) A_i, φ sont des fonctions de x_i.

Pour les interactions faibles ces conditions peuvent être remplies dans une bonne approximation [3].

$L\,dt$ est un invariant, et pour les interactions gravitationnelles faibles la durée dans laquelle le mouvement peut être considéré comme rectiligne et uniforme, est suffisamment grande, par conséquent la condition 1° est remplie en bonne approximation.

Dans l'approximation newtonienne seul α_0 diffère de zéro, donc les fonctions A_i et $\varphi - (\lambda m_0 \alpha_0 c^2/2r)$ sont des termes correctifs. En appliquant la méthode de Hamilton-Jacobi, nous pouvons donner les fonctions de vitesse $v_i^0 = v_i^0(x_j)$ saitsfaisant aux équations

$$\frac{1}{2} m_0 v^2 - \frac{K m_0 M_0}{r} = E, \qquad r^2 \dot{\Theta} = b,$$

où E, b sont des constantes ayant respectivement le caractère d'énergie et de vitesse aréolaire. Ces fonctions introduites dans les termes correctifs, nous obtenons en bonne approximation les fonctions A_i, φ figurant dans la 2° condition.

Notre procédure est encore justifiée par ce qui suit. En utilisant la fonction L nous obtenons pour les composants de la force

$$(6) \qquad K_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0 v_i}{\sqrt{1 - v^2/c^2}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{x}_i}\right) - \frac{\partial V}{\partial x_i} = \frac{m_0 \lambda c^2 x_i}{r^3}\left[\frac{\alpha_0}{2} + \left(\alpha_1 - \frac{\alpha_0^2}{4} - \frac{\alpha_0}{2} + \beta_0\right)\frac{\lambda}{r} + \right.$$

$$\left. + \frac{3(\beta_0 - \gamma_0)}{2c^2}\,\dot{r}^2 + \left(\frac{\alpha_0}{2} - 2\beta_0 + \gamma_0\right)\frac{v^2}{2c^2}\right] + \left(\gamma_0 - \frac{\alpha_0}{2}\right)\frac{\lambda m_0 \dot{r}\dot{x}_i}{r^2}.$$

Mais

$$K_i = -\frac{\partial \varphi}{\partial x_i} + \dot{x}_k\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right)$$

nous conduit à des fonctions (6). Il faut tenir compte dans les calculs de ce qui suit: on donne A_i et φ en fonction de x_i, E, b. Après la differentiation — et après elle seulement — E_0, b peuvent être substitués par leurs expressions.

Nous avons encore

$$(7) \qquad L_{\text{Sch}}\,\mathrm{d}t - L_{\text{F}}\,\mathrm{d}t = \lambda\left(-\frac{m_0}{r}v^2 + \frac{m_0}{r}\dot{r}^2 + \frac{\lambda m_0 c^2}{r^2}\right)\mathrm{d}t = -\,\mathrm{d}\left(\frac{m_0\lambda x_i \dot{x}_i^0}{r}\right),$$

(où L_{Sch}, L_{F} sont les fonctions établies à l'aide des expressions $\mathrm{d}s^2$ données respectivement pas SCHWARZSCHILD et FOK), et pour les coordonnées harmoniques

$$(8) \qquad P_i = -\frac{m_0 v_i}{\sqrt{1 - v^2/c^2}} + A_i = \frac{M_0^* v_i}{\sqrt{1 - v^2/c^2}}, \qquad \text{où} \qquad M_0^* = m_0\left(1 + \frac{3\lambda}{r}\right).$$

Conclusions:

a) l'interaction gravitationnelle d'un point matériel en mouvement et d'un point matériel en repos peut être caractérisée, pour le cas des interactions faibles, par un quadri-vecteur-potentiel;

b) les fonctions de Lagrange utilisées sont équivalentes, par conséquent sont équivalents aussi les quadri-vecteurs-potentiels établis grâce à ces fonctions;

c) l'interaction du point matériel et du champ a pour conséquence l'augmentation de la masse de repos.

2. – Interaction gravitationnelle des points en movement.

Nous partons de l'hypothèse que le problème étudié est géométrisable. Dans ce cas L ne contient que r (distance des points matériels), $V_{i\alpha}$ (composant de la vitesse α pour le point matériel i; ($\alpha = 1, 2, 3$, $i = 1, 2$). Pour le cas d'un point en mouvement et un autre point en repos les effects relativistes sont déterminés par les termes λ/r et v^2/c^2, donc pour deux corps en mouvement nous avons supposé:

1) L est une fonction de

$$\frac{\lambda_1}{r}, \quad \frac{\lambda_2}{r}, \quad \frac{v_{i\alpha}v_{k\beta}}{c^2},$$

donc

$$L = L\left(\frac{\lambda_1}{r}, \frac{\lambda_2}{r}, \frac{v_{i\alpha}v_{k\beta}}{c^2}\right),$$

où λ_1, λ_2 sont les rayons gravifiques des particules, et encore

2) pour des valeurs de r très grandes

$$\lim_{r\to\infty} L = -\sum_i m_{0i}c^2\left(1 - \frac{v_i^2}{c^2}\right) \tag{9}$$

3) L est une fonction des invariants euclidiens [7].

4) L est invariant pour un changement des indices 1 et 2.

Développons la fonction L en série au voisinage de la valeur zéro des variables indépendantes. En nous limitant aux termes d'ordre de grandeur m_0c^2, m_0v^2, m_0v^4/c^2, nous obtenons:

$$L = -\sum_i m_{0i}c^2\sqrt{1 - \frac{v_i^2}{c^2} + \frac{A}{r} + \frac{B}{c^2r^2} + \frac{C_{ik\alpha\beta}v_{i\alpha}v_{k\beta}}{c^2r}}, \tag{10}$$

où A, B, $C_{ik\alpha\beta}$ sont des constantes.

Voyons les conséquences de l'hypothèse 3). Pour le cas des deux points matériels nous aovns les invariants euclidiens:

$$r, \quad v_1^2 = v_{1\alpha}v_{1\alpha}, \quad v_2^2 = v_{2\alpha}v_{2\alpha}, \quad v_{1\alpha}^0 v_{2\alpha}^0 = \cos\varepsilon, \quad r_\alpha^0 v_{1\alpha}^0 = \cos\Theta_1, \quad r_\alpha^0 v_{2\alpha}^0 = \cos\Theta_2,$$

avec

$$r_\alpha^0 = \frac{x_{1\alpha} - x_{2\alpha}}{r}, \qquad v_{i\alpha}^0 = \frac{v_{i\alpha}}{v_i}.$$

Le dernier terme dans (10) est une fonction homogène du second ordre des composants de la vitesse, par conséquent il est une composition linéaire des expressions d'invariants:

$$v_{1\alpha}v_{1\alpha} = v_1^2, \qquad v_{1\alpha}v_{2\alpha} = v_1v_2\cos\varepsilon, \qquad v_{2\alpha}v_{2\alpha} = v_2^2,$$
$$(v_{1\alpha}r_\alpha^0)^2 = v_{1r}^2, \qquad v_{1\alpha}r_\alpha^0 v_{2\beta}r_\beta^0 = v_{1r}v_{2r}, \qquad (v_{2\alpha}r_\alpha^0)^2 = v_{2r}^2.$$

Il résulte de la condition 4) que les coefficients de v_1^2 et v_2^2, respectivement v_{1r}^2 et v_{2r}^2 sont égaux.

Donc, la fonction de Lagrange satisfaisant aux conditions ci-dessus sera:

$$L = -\sum_i m_{0i}c^2\sqrt{1 - \frac{v_i^2}{c^2} + \frac{A}{r} + \frac{B}{c^2r^2} + \frac{1}{c^2r}\left[C(v_1^2 + v_2^2) + Dv_{1\alpha}v_{2\alpha} + E(v_{1r}^2 + v_{2r}^2) + Fv_{1r}v_{2r}\right]}.$$

Pour déterminer les constantes $A, \ldots, F$ examinons des cas particuliers.

A) Une partie des la fonction d'interaction reflète des interactions pour lesquelles est valable le principe de la superposition. Cette partie a la forme:

$$\frac{A_0}{r} + \frac{1}{c^2 r}\left(D_0 v_{1\alpha} v_{1\alpha} + F_0 v_{1r} v_{2r}\right).$$

Soient les vitesses des particules égales: $v_{1\alpha} = v_{2\alpha} = v_\alpha$. Vu qu'ont lieu des interactions s'intégrant dans la théorie restreinte de la relativité, la fonction d'interaction s'écrira comme suit:

$$A^*_\alpha v_\alpha - \varphi^*.$$

Dans le système $\bar{K}$ lié aux points matériels en mouvement

$$\bar{A}^*_\alpha = 0, \qquad \bar{\varphi}^* = -\frac{K m_0 M_0}{\bar{r}}.$$

$\bar{r}$ se confond avec la distance des points dans le système $\bar{K}$. En appliquant les formules de transformation de Lorentz pour le quadri-vecteur aux composants A^*_α, $A^*_4 = (i/c)\varphi^*$, nous obtiendrons pour le système K:

$$A^*_\alpha = -\beta \frac{K m_{01} m_{02}}{c^2 \bar{r}} v_\alpha, \qquad A^*_4 = \frac{i}{c}\varphi^* = -i\beta \frac{K m_{01} m_{02}}{c^2 \bar{r}}, \quad \text{avec} \quad \beta = \frac{1}{\sqrt{1 - v^2/c^2}},$$

et

$$\bar{r} = r\left(1 + \frac{v_r^2}{2c^2}\right) + \ldots,$$

r étant la distance des points dans le système K et $v_r = v^0_\alpha r_\alpha$. Donc

$$A^*_\alpha v_\alpha - \varphi^* = \frac{K m_{01} m_{02}}{r}\left(1 - \frac{v^2}{2c^2} - \frac{v_r^2}{2c^2}\right).$$

Nous avons donc obtenu pour la fonction d'interaction

$$\frac{K m_{01} m_{02}}{r}\left(1 - \frac{1}{2c^2} v_{1\alpha} v_{2\alpha} - \frac{1}{2c^2} v_{1r} v_{2r}\right), \tag{11}$$

qui est le correspondant gravitationnel de la formule de Breit.

B) Dans ce qui suit nous établirons la fonction d'interaction s'intégrant dans une théorie non-linéaire.

Dans le cas $m_{01} \ll m_{02}$ v_1 est négligeable, dans le cas $m_{02} \ll m_{01}$ c'est v_2 qui est négligeable. Nous avons déjà établi la fonction L pour ces cas particuliers (3). En tenant compte du fait que dans les cas particuliers r_1 et r_2 représentent des vitesses relatives, nous aurons, dans le cas des coordonnées harmoniques et ordinaires respectivement

$$(12) \qquad -\frac{K m_{01} m_{02}(m_{01}+m_{02})}{2c^2 r^2} + \frac{3K m_{01} m_{02}}{2c^2 r}(v_{1\alpha}-v_{2\alpha})(v_{1\alpha}-v_{2\alpha}),$$

et

$$(13) \qquad \frac{K m_{01} m_{02}(m_{01}+m_{02})}{2c^2 r^2} + \frac{K m_{01} m_{02}}{2c^2 r}\left[(v_{1\alpha}-v_{2\alpha})(v_{1\alpha}-v_{2\alpha}) + 2(v_{1r}-v_{2r})^2\right].$$

Nous aurons encore pour la différence de ces expressions (3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{K m_{01} m_{02}(x_{1\alpha}-x_{2\alpha})(v_{1\alpha}-v_{2\alpha})}{c^2 r}\right].$$

La somme des expressions (9), (11), (13) respectivement (9), (11), (12) nous conduit à la fonction de Lagrange donnée respectivement pour le cas des coordonnées ordinaires et harmoniques. Pour les coordonnées harmoniques nous avons retrouvé la fonction de I. G. FICHTENHOLTZ.

Avant de commencer l'étude des systèmes de points, récapitulons les résultats obtenus:

a) les fonctions L données pour l'intéraction des deux points matériels sont équivalentes. Il est convenable d'utiliser celle, pour laquelle nous obtenons l'expression la plus simple: (12) résultant pour les coordonnées harmoniques;

b) en appliquant notre procédure, l'on peut obtenir d'une manière très simple des approximations du troisième ordre;

c) concernant la structure de la fonction L nous avons obtenu ce qui suit: la fonction L se compose de trois termes ($L = L_p + L_{pc} + L_c$). L_p est lié au mouvement mécanique des particules, L_{pc} exprime l'intéraction des particules avec le champ gravitationnel, L_c est le terme propre au champ. La fonction d'interaction ($L_{pc} + L_c$) se compose des termes s'intégrant dans une théorie linéaire et d'autre part dans une théorie non-linéaire;

d) pour la signification physique des termes entrant dans la fonction d'interaction nous possédons les données suivantes:

Dans la partie qui reflète des interactions s'intégrant dans les théories linéaires, à côté du potentiel newtonien figurent des termes exprimant l'in-

teraction des courants gravitationnels (la masse m_0 représente une charge gravifique $\sqrt{\kappa}\, m_0$). L'apparition de ces termes témoigne de l'existence d'interactions gravitationnelles de type électrique tout aussi bien que de type magnétique ([1,3]).

L'expression s'intégrant dans la théorie non-linéaire reflète deux effets. Le terme purement statique exprime l'autoaction du champ, tandis que le terme dynamique exprime le fait que le déplacement relatif des particules est entravé partiellement (ceci a pour conséquence l'augmentation de la masse de repos, donc l'apparition d'une masse gravitationnelle).

Ci-dessous nous allons établir la fonction L pour le système de n points matériels, tout en nous servant des résultats obtenus.

Concernant l'expression L_p nous avons

$$L_p = -\sum_i m_{0i} c^2 \sqrt{1 - \frac{v_i^2}{c^2}}. \tag{14}$$

Pour déterminer L_c nous procédons de la manière suivante. Les points soient en repos, à l'exception du point i. Dans ce cas-ci nous aurons pour le point i

$$\mathrm{d}s_i^2 = c^2 \left(1 - 2\frac{\varphi_i}{c^2} + 2\frac{\varphi_i^2}{c^4}\right) \mathrm{d}t^2 - \left(1 + 2\frac{\varphi_i}{c^2}\right) v_i^2 \, \mathrm{d}t^2 ,$$

où

$$\varphi_i = \sum_k{}' \frac{K m_{0k}}{r_{ik}} \qquad \text{avec} \qquad r_{ik} = |\boldsymbol{r}_i - \boldsymbol{r}_k| .$$

Etant donné que

$$L_i \, \mathrm{d}t = -m_{0i} c \, \mathrm{d}s_i$$

nous avons

$$(L_c)_i = m_{0i}\varphi_i - m_{0i}\frac{\varphi_i^2}{2c^2} = \sum_k{}' \frac{K m_{0i} m_{0k}}{r_{ik}} - \frac{m_{0i}}{2c^2}\left(\sum_k{}' \frac{K m_{0k}}{r_{ik}}\right)^2,$$

donc

$$L_c = \frac{1}{2}\sum_{i,k} \frac{K m_{0i} m_{0k}}{r_{ik}} - \frac{1}{2c^2}\sum_i m_{0i}\left(\sum_k{}' \frac{K m_{0k}}{r_{ik}}\right)^2. \tag{15}$$

Tenant compte des expressions (11), (12) établies pour le cas des deux point matériels, nous aurons:

$$L_{1c} = \frac{3}{4c^2}\sum_{i,k} \frac{K m_{0i} m_{0k}}{r_{ik}} (\boldsymbol{v}_i - \boldsymbol{v}_k)^2 - \tag{16}$$

$$- \frac{1}{4c^2}\sum_{i,k} \frac{K m_{0i} m_{0k}}{r_{ik}} \left[(\boldsymbol{v}_i, \boldsymbol{v}_k) + (\boldsymbol{v}_i, \boldsymbol{n}_{ik})(\boldsymbol{v}_k, \boldsymbol{n}_{ik})\right] \qquad \text{où} \qquad \boldsymbol{n}_{ik} = \frac{\boldsymbol{r}_i - \boldsymbol{r}_k}{r_{ik}}.$$

La somme des expressions (14), (15), (16) nous conduit à la fonction de I. G. Fichtenholtz.

3. Interaction gravitationnelle des corps matériels en rotation.

Sous ce point nous allons examiner l'interaction de deux corps en rotation. Les résultats developpés ci-dessous sont ceux mentionnées dans l'introduction (*c*), *d*), *e*)).

En nous servant des résultats établis, nous allons démontrer qu'en partant de la fonction de Lagrange de I. G. FICHTENHOLTZ nous pouvons donner la fonction de V. A. FOK par une méthode directe, sans que le théorème du moment de la quantité de mouvement soit utilisé.

En effet, nous avons deux systèmes de points peuvant être considérés comme étant en mouvement fixé, pour une durée suffisamment longue. Introduisons les notations suivantes: $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ – vecteurs de vitesse angulaire; $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ – vecteurs de position pour les centres des sphères 1, respectivement 2; $\boldsymbol{r}_1$ est le vecteur d'un point de la sphère 1 mesuré du centre de la sphère; $\boldsymbol{r}_2$ est le même vecteur pour la sphère 2. Soient encore $\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2$, $R = |\boldsymbol{R}|$, $\boldsymbol{R}_{12} = \boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}_1$. Nous supposons que les fonctions de la densité sont

$$\mu_i = \mu_{0i} - \alpha_i \frac{r_i^2}{a_i^2},$$

(μ_{0i}, α_i sont des constantes, a_i est un rayon sphérique), et que les rayons gravitationnels des sphères sont plus petits que leurs rayons géométriques: $\lambda_i = (Km_{0i}/c^2) \ll a_i$.

En utilisant la fonction de I. G. FICHTENHOLTZ, nous avons

$$L_p = -\sum_i \int \mu_i \sqrt{c^2 - v_i^2}\, \mathrm{d}\tau_i\,, \qquad \text{avec} \qquad \boldsymbol{v}_i = \boldsymbol{v}_{i0} + (\boldsymbol{\omega}_i \times \boldsymbol{r}_i)\,, \tag{17}$$

où $\boldsymbol{v}_{i0}$ est la vitesse de translation pour le centre de sphère i, et

$$L_c = K \iint \frac{\mu_1 \mu_2\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2}{|\boldsymbol{R} + \boldsymbol{r}_2 - \boldsymbol{r}_1|} - \frac{K^2}{2c^2}\left[m_{02}^2 \int \frac{\mu_1\, \mathrm{d}\tau_1}{|\boldsymbol{R} - \boldsymbol{r}_1|} + m_{01}^2 \int \frac{\mu_2\, \mathrm{d}\tau_2}{|\boldsymbol{R} + \boldsymbol{r}_2|} \right] + \dots\,, \tag{18}$$

(pour L_c nous avons négligé des termes à valeurs insignifiantes dans le cas $\lambda_i \ll a_i$).

$$L_c = \iint \frac{K \mu_1 \mu_2}{2c^2 R_{12}} \left[3r_1^2 + 3r_2^2 - 7(\boldsymbol{v}_1, \boldsymbol{v}_2) - \frac{(\boldsymbol{v}_1, \boldsymbol{R}_{12})(\boldsymbol{v}_2, \boldsymbol{R}_{12})}{R_{12}^2} \right] \mathrm{d}\tau_1\, \mathrm{d}\tau_2 = \tag{19}$$

$$= L_{pc}^{(t)} + L_{pc}^{(rt)} + L_{rc}^{(r)}\,.$$

$L_{pc}^{(t)}$ est un terme de translation pure, $L_{pc}^{(rt)}$ est un terme de rotation-translation, $L_{pc}^{(r)}$ représente un terme de rotation pure.

Les expressions obtenues peuvent être appliquées aux corps d'une forme quelconque. En appliquant les résultats à l'intéraction de deux sphères en rotation, nous obtenons les expressions appliquables pour le cas où la distance des centres est d'un ordre de grandeur plus grande que les rayons des sphères:

$$(20)\qquad L_p = \sum_i \left[- m_{0i}c^2 + \frac{m_{0i}}{2} v_{i0}^2 + \frac{\Theta_i}{2}\omega_i^2 + \frac{m_{0i}}{8c^2} v_{i0}^4 + \frac{\Theta_i}{2c^2} v_{i0}^2 \omega_i^2 - \right.$$

$$\left. - \frac{\Theta_i}{4c^2} (\boldsymbol{v}_{i0}, \boldsymbol{\omega}_i)^2 + \frac{V_i a_i^4 \omega_i^4}{35c^2} \left(\mu_{0i} - \frac{7}{9}\alpha_i \right) \right] + \dots ,$$

V_i est le volume de la sphère i, Θ_i le moment d'inertie du corps i pour un axe passant par le centre.

$$\Theta_i = \frac{2V_i a_i^2}{5} \left(\mu_{0i} - \frac{5}{7}\alpha_i \right) ;$$

$$(21)\qquad L_c = - \frac{K m_{01} m_{02}}{R} - \frac{K^2}{2c^2} \frac{m_{01} m_{02}^2 + m_{01}^2 m_{02}}{R^2} -$$

$$- \frac{K^2}{70c^2} \frac{m_{02}^2 V_1 a_1^2 (7\mu_{01} - 5\alpha_1) + m_{01}^2 V_2 a_2^2 (7\mu_{02} - 5\alpha_2)}{R^4} + \dots ,$$

$$(22)\qquad L_{pc}^{(t)} = - \frac{K m_{01} m_{02}}{2c^2 R} \left[3v_{10}^2 + 3v_{20}^2 - 7(\boldsymbol{v}_{10}, \boldsymbol{v}_{20}) - \frac{(\boldsymbol{v}_{10}, \boldsymbol{R})(\boldsymbol{v}_{20}, \boldsymbol{R})}{R^2} \right] - \frac{K(\boldsymbol{v}_{10}, \boldsymbol{v}_{20})}{4c^2 R^3} \cdot$$

$$\cdot \left(m_{01}\Theta_2 + m_{02}\Theta_1 - \frac{3\Theta_1\Theta_2}{R^2} \right) + \frac{3K(\boldsymbol{v}_{10}, \boldsymbol{R})(\boldsymbol{v}_{20}, \boldsymbol{R})}{4c^2 R^5} \left(m_{01}\Theta_2 + m_{02}\Theta_1 - \frac{5\Theta_1\Theta_2}{R^2} \right) + \dots ,$$

$$(23)\qquad L_{pc}^{(rt)} = \frac{K m_{02}\Theta_1}{2c^2 R^3} (\boldsymbol{\omega}_1, \boldsymbol{R}, 3\boldsymbol{v}_{10} - 4\boldsymbol{v}_{20}) - \frac{K m_{01}\Theta_2}{2c^2 R^3} (\boldsymbol{\omega}_2, \boldsymbol{R}, 3\boldsymbol{v}_{20} - 4\boldsymbol{v}_{10}) ,$$

$$(24)\qquad L_{pc}^{(r)} = \frac{K\Theta_1\Theta_2}{c^2 R^3} \left[(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) - \frac{3(\boldsymbol{\omega}_1, \boldsymbol{R})(\boldsymbol{\omega}_2, \boldsymbol{R})}{R^2} \right] + \frac{3K}{2c^2} \frac{m_{02}\Theta_1\omega_1^2 + m_{01}\Theta_2\omega_2^2}{R} .$$

Les fonctions d'interaction ci-dessus contiennent des termes s'intégrant dans une théorie linéaire (par exemple le terme exprimant l'interaction des courants gravifiques), et des termes s'intégrant dans les théories non-linéaires (autoaction du champ, autoaction des corps ayant lieu par le champ).

Les résultats obtenus sont en concordance avec les résultats de V. A. Fok. Nous avons complété ces derniers — dans le cas de deux sphères — avec le

terme concernant l'interaction des moments gravifiques. Ce moment est donné par l'hypothèse

$$\boldsymbol{M}_{\mathrm{gr}} = \frac{\sqrt{k}}{2c}\,\boldsymbol{M}_{\mathrm{mech}} = \frac{\sqrt{k}}{2c}\,\Theta\boldsymbol{\omega}\,, \tag{25}$$

où $\boldsymbol{M}_{\mathrm{mech}}$ est le moment de la quantité du mouvement. Le terme concernant l'interaction des moments gravifiques prend la forme:

$$4\,\frac{(\boldsymbol{M}_{\mathrm{gr}}^{(1)}, \boldsymbol{M}_{\mathrm{gr}}^{(2)})R^2 - 3(\boldsymbol{M}_{\mathrm{gr}}^{(1)}, \boldsymbol{R})(\boldsymbol{M}_{\mathrm{gr}}^{(2)}, \boldsymbol{R})}{R^5}\,. \tag{26}$$

Cette expression est analogue à celle concernant l'interaction des moments magnétiques (l'analogie sera encore plus parfaite si nous ne tenons compte que de la partie due à des interactions linéaires, égale à la quatrième partie de l'expression (26).

* * *

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RIASSUNTO (*)

Si danno alcuni risultati concernenti la struttura e i legami delle funzioni di Lagrange che sono state date per il caso d'interazione gravitazionale dei punti e dei corpi materiali. Nel caso di due sfere rigide in rotazione (per le quali la distribuzione della densità è data dalla formula di Roche) abbiamo ricavato la funzione L, che si applica nel caso in cui la distanza dei centri delle sfere è almeno di un ordine di grandezza maggiore della lunghezza dei raggi delle sfere. Si introduce la nozione del momento gravitazionale dimostrando che per l'interazione dei momenti gravitazionali sono valide espressioni analoghe a quelle per l'interazione dei momenti magnetici.

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Dispersion Relations and Vibrational Frequency Spectra (*).

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Summary. — Many relations have been given connecting the dispersion relations and frequency spectrum for a vibrating lattice. It is the purpose of this paper to show that these relations can all be derived from a single integral representation by using different representations of the δ-function.

One of the most fundamental problems in the theory of lattice dynamics is the determination of the frequency distribution function or frequency spectrum from the dispersion relations. The dispersion relations express the normal modes of a crystal in terms of the components of the wave vector for the propagation of elastic waves through the crystal. This type of problem can be stated in quite general mathematical terms and also arises in many other situations in statistical mechanics. Several methods for finding the frequency spectrum from the dispersion relations have been proposed and used in various problems; however, the derivations given are of different forms making it difficult to see a unifying relationship between them. It is our purpose in this paper to show how these different expressions for the frequency spectrum in terms of the dispersion relations can be found as the several different forms of the same basic formula. Since most of the formulae to be presented here have already been given by other authors, we cannot claim that the specific results are new, but rather that the general method provides a convenient way of looking at such problems. Although it is not always possible to find the

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frequency spectrum explicitly, it is very often the case that information about singularities and other qualitative features of the spectrum can be obtained by various approximate means from the formulae to follow.

The usual derivation of the dispersion relations proceeds from the equations of motion of the atoms in the lattice [1]

$$M_k \ddot{u}_\alpha \begin{pmatrix} l \\ k \end{pmatrix} = -\sum_{\beta, l', k'} \Phi_{\alpha\beta} \begin{pmatrix} l & l' \\ k & k' \end{pmatrix} u_\beta \begin{pmatrix} l' \\ k' \end{pmatrix}, \tag{1}$$

where $u_\alpha \begin{pmatrix} l \\ k \end{pmatrix}$ is the α-component of the displacement of the k-th atom in the l-th unit cell from its equilibrium position; M_k is the mass of the k-th kind of atom; and the $\Phi_{\alpha\beta} \begin{pmatrix} l\ l' \\ k\ k' \end{pmatrix}$ are the general force constants of the lattice which due to lattice periodicity depend on l and l' only through the difference $\bar{l} = l - l'$. A running wave solution of the form

$$u_\alpha \begin{pmatrix} l \\ k \end{pmatrix} = \frac{u_{\alpha k}^{(0)}}{\sqrt{M_k}} \exp\left[i\omega t - 2\pi i \boldsymbol{y} \cdot \boldsymbol{x}(l)\right], \tag{2}$$

transforms eq. (1) into

$$\omega^2 u_{\alpha k}^{(0)} = \sum_{\beta k'} C_{\alpha\beta} \begin{pmatrix} \boldsymbol{y} \\ k \quad k' \end{pmatrix} u_{\beta k'}^{(0)}, \tag{3}$$

where

$$C_{\alpha\beta} \begin{pmatrix} \boldsymbol{y} \\ k \quad k' \end{pmatrix} = \frac{1}{\sqrt{M_k M_k'}} \sum_{\bar{l}} \Phi_{\alpha\beta} \begin{pmatrix} \bar{l} \\ k \quad k' \end{pmatrix} \exp\left[2\pi i \boldsymbol{y} \cdot \boldsymbol{x}(l)\right]. \tag{4}$$

In eqs. (2) and (3) the vector $\boldsymbol{x}(l)$ is the position vector of the l-th unit cell

$$\boldsymbol{x}(l) = l_1 \boldsymbol{a}_1 + l_2 \boldsymbol{a}_2 + l_3 \boldsymbol{a}_3, \tag{5}$$

where the $\boldsymbol{a}_i$ are the primitive translation vectors of the crystal and the $\{l_i\}$ are integers. We assume, following BORN and VON KARMAN, that the displacements $u_\alpha \begin{pmatrix} l \\ k \end{pmatrix}$ are periodic in a macrocrystal containing N^3 unit cells whose shape is the same as that of the primitive cell of the crystal and whose edges are defined by the vectors $N\boldsymbol{a}_1$, $N\boldsymbol{a}_2$, $N\boldsymbol{a}_3$. Equation (4) then tells us that the vector $\boldsymbol{y}$ is defined by

$$\boldsymbol{y} \cdot \boldsymbol{a}_j = \frac{n_j}{N}, \qquad (j = 1, 2, 3), \tag{6}$$

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[1] M. BORN and K. HUANG: *Dynamical Theory of Crystal Lattices*, 1st ed. (London, 1954).

where n_j are integers. This result implies that the allowed values of the vector $\boldsymbol{y}$ are given by:

$$\boldsymbol{y} = \frac{1}{N}(n_1\boldsymbol{b}^1 + n_2\boldsymbol{b}^2 + n_3\boldsymbol{b}^3), \tag{7}$$

where the $\boldsymbol{b}^j$ are the primitive translation vectors of the reciprocal lattice and are defined by

$$\boldsymbol{a}_i \cdot \boldsymbol{b}^j = \delta_{ij}\,. \tag{8}$$

Since adding any multiple of $\boldsymbol{b}^j$ $(j = 1, 2, 3)$ to $\boldsymbol{y}$ leaves $C_{\alpha\beta}\begin{pmatrix} \boldsymbol{y} \\ k\ k' \end{pmatrix}$ unchanged, we see that all distinct values of ω^2 are obtained by restricting y to lie in a single unit cell in reciprocal space. This result in turn implies that the integers n_j take on the values 0, 1, 2, ..., $N-1$. It is easily seen that this procedure leads to the correct number of degrees of freedom for the lattice. The coefficients $C_{\alpha\beta}\begin{pmatrix} \boldsymbol{y} \\ k\ k' \end{pmatrix}$ become

$$C_{\alpha\beta}\begin{pmatrix} \boldsymbol{\theta} \\ k\quad k' \end{pmatrix} = \frac{1}{\sqrt{M_k M_{k'}}} \sum_{l_1 l_2 l_3} \Phi_{\alpha\beta}\begin{pmatrix} \boldsymbol{l} \\ k\quad k' \end{pmatrix} \exp\left[i\boldsymbol{l}\cdot\boldsymbol{\theta}\right], \tag{9}$$

where

$$\boldsymbol{l} = (l_1, l_2, l_3)\,, \qquad \boldsymbol{\theta} = \frac{2\pi}{N}(n_1, n_2, n_3)\,.$$

The allowed values of the vector $\boldsymbol{\theta}$ are distributed throughout the cube two of whose diagonally opposite corners are located at (0, 0, 0) and $(2\pi, 2\pi, 2\pi)$ with a uniform density of $N^3/(2\pi)^3$.

This discussion is quite general. In some cases, most notably for body-centered and face-centered cubic lattices, a more convenient choice for the lattice translation vector $\boldsymbol{x}(l)$ is

$$\boldsymbol{x}(l) = \frac{a_0}{2}(l_1, l_2, l_3)\,,$$

where a_0 is the lattice parameter and l_1, l_2, l_3 are all even or all odd integers for the b.c.c. case, and three integers whose sum is even for the f.c.c. case. For these two cases the results to be derived are still valid.

From eq. (3) we see that the normal mode frequencies are the roots of the $3s \times 3s$ $(k = 1, 2, ..., s)$ secular determinant:

$$D(\omega^2) = \det\left\{C_{\alpha\beta}\begin{pmatrix} \boldsymbol{y} \\ k\quad k' \end{pmatrix} - \omega^2\delta_{\alpha\beta}\delta_{kk'}\right\} = 0\,. \tag{10}$$

The $3s$ solutions to this equation may be written formally as

$$\omega^2 = F_j(\theta_1, \theta_2, \theta_3) \qquad (j = 1, 2, \ldots, 3s) \tag{11}$$

and constitute the $3s$ dispersion relations for the lattice corresponding to the $3s$ branches of the frequency spectrum. In what follows we consider one particular branch, except where stated otherwise, and ask the following question: If $G_j(\omega^2)\,\mathrm{d}\omega^2$ is the fraction of normal modes with frequencies in the interval $(\omega^2, \omega^2 + \mathrm{d}\omega^2)$ for the j-th branch, how can we find $G_j(\omega^2)$ if we are given the function $F_j(\theta_1, \theta_2, \theta_3)$? The relation between $G_j(\omega^2)$ and the distribution function for the normal mode frequencies $g_j(\omega)$ is $g_j(\omega) = 2\omega G_j(\omega^2)$.

The basic relationship between $G_j(\omega^2)$ and $F_j(\theta_1, \theta_2, \theta_3)$ has been given by Bowers and Rosenstock [2]

$$G_j(\omega^2) = \frac{1}{\pi^3} \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \delta[\omega^2 - F_j(\theta_1, \theta_2, \theta_3)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 \,. \tag{12}$$

Expressed in this form $G_j(\omega^2)$ is normalized to unity. Montroll [3] has utilized the representation obtained when one uses the formula

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\,[-ixy]\,\mathrm{d}y \,. \tag{13}$$

With an interchange of the orders of integration we find that

$$G_j(\omega^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\,[-iy\omega^2]\,\mathrm{d}y \left\{ \frac{1}{\pi^3} \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \exp\,[iyF_j(\theta_1, \theta_2, \theta_3)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 \right\}, \tag{14}$$

i.e., the frequency spectrum is proportional to the Fourier transform of the function

$$f_j(y) = \frac{1}{\pi^3} \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \exp\,[iyF_j(\theta_1, \theta_2, \theta_3)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 \,. \tag{15}$$

This form for $G_j(\omega^2)$ has recently [4] been shown to be particularly useful as a basis for a unified discussion of the singularities in the frequency spectrum

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[2] W. A. Bowers and H. B. Rosenstock: *Journ. Chem. Phys.*, **18**, 1056 (1950).

[3] E. W. Montroll: *Proc. Third Berkeley Symposium on Mathematical Statistics and Probability*, 1st ed. (Berkeley, 1956), p. 209.

[4] A. Maradudin and J. Peretti: *Comp. Ren.*, **247**, 2310 (1958).

associated with analytic critical points in the surfaces of constant frequency in $\boldsymbol{\theta}$-space.

Use of the Laplace transform form for $\delta(x)$

$$\delta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\,[xy]\,\mathrm{d}y\,, \tag{16}$$

leads to the equivalent expression for $G_j(\omega^2)$

$$G_j(\omega^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\,[y\omega^2] f_j(y)\,\mathrm{d}y, \tag{17}$$

where

$$f_j(y) = \frac{1}{\pi^3}\int_0^{\pi}\!\!\int\!\!\int \exp\,[-\,yF_j(\theta_1,\,\theta_2,\,\theta_3)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3\,. \tag{18}$$

Another representation which has been found useful in theoretical investigations, due, in different forms to DYSON [5] and PERETTI [6] can be obtained by using the identity

$$\delta(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\varepsilon^2 + x^2} = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\mathrm{Im}\,\frac{1}{x - i\varepsilon} = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\mathrm{Im}\,\frac{\mathrm{d}}{\mathrm{d}x}\log\,(x - i\varepsilon)\,. \tag{19}$$

If changes in limiting processes are freely allowed we find

$$G_j(\omega^2) = \frac{1}{\pi^1}\lim_{\varepsilon\to 0+}\mathrm{Im}\int_0^{\pi}\!\!\int\!\!\int \frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3}{\omega^2 - i\varepsilon - F_j(\theta_1,\,\theta_2,\,\theta_3)} = \tag{20}$$

$$= \frac{1}{\pi^4}\lim_{\varepsilon\to 0+}\mathrm{Im}\,\frac{\mathrm{d}}{\mathrm{d}\omega^2}\int_0^{\pi}\!\!\int\!\!\int \log\,(\omega^2 - i\varepsilon - F_j(\theta_1,\,\theta_2,\,\theta_3))\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3\,.$$

This last integral can be put into a more suggestive form. Suppose for the moment that we return to the discrete problem of a lattice containing N^3 unit cells. The secular determinant $D(\omega^2)$ can clearly be written as

$$D(\omega^2) = \prod_{j=1}^{3s}\prod_{\theta_1,\,\theta_2,\,\theta_3}(\omega^2 - F_j(\theta_1,\,\theta_2,\,\theta_3))\,, \tag{21}$$

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(5) F. J. DYSON: *Phys. Rev.*, **92**, 1331 (1953).

(6) J. PERETTI: *Journ. Chem. Phys. of Solids* (to appear).

so that

$$\log D(\omega^2) = \frac{N^3}{\pi^3} \sum_{j=1}^{3s} \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \log\left[\omega^2 - F_j(\theta_1, \theta_2, \theta_3)\right] \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 . \tag{22}$$

Hence, if we know the function $D(\omega^2)$ we see from eq. (20) that

$$G(\omega^2) = \sum_{j=1}^{3s} G_j(\omega^2) = \frac{1}{\pi N^3} \lim_{\varepsilon \to 0+} \operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}\omega^2} \log D(\omega^2 - i\varepsilon) . \tag{23}$$

PERETTI [6] has recently emphasized the usefulness of the function $F(z) = -(1/N^3)(\mathrm{d}/\mathrm{d}z) \log D(z - i\varepsilon)$ in discussing singularities in frequency spectra. The function $D(z)$, as a function of the complex variable z can be shown to be defined everywhere in the z-plane except for a branch cut connecting $-\omega_L$ to ω_L along the real axis, where ω_L is the maximum allowable frequency for the lattice.

The result given in eq. (23) can be proved more rigorously by means of Hilbert [7] or Stieltjes [8] transforms. It can be shown that the expression in eq. (20) is related to the Lippmann-Schwinger formulation of scattering problems.

Still another representation for $G_j(\omega^2)$ can be obtained by expanding the δ-functions in terms of a complete orthonormal set of functions $\{\varphi_n(x)\}$ with the aid of the relation

$$\delta(x - y) = \sum_n \varphi_n(x) \varphi_n(y) \tag{24}$$

which implies that

$$G_j(\omega^2) = \frac{1}{\pi^3} \sum_{n=0}^{\infty} \varphi_n(\omega^2) \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \varphi_n[F_j(\theta_1, \theta_2, \theta_3)] \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 . \tag{25}$$

This type of result has been extensively used in connection with the method of moments [9]. If we take our orthonormal functions to be the Legendre polynomials we obtain the result that

$$G_j(\omega^2) = \frac{1}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{2n+1} P_n(\omega^2) \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} P_n[F_j(\theta_1, \theta_2, \theta_3)] \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 . \tag{26}$$

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[7] E. C. TICHMARSH: *Introduction to the Theory of Fourier Integrals*, 1st ed. (London, 1937).

[8] D. WIDDER: *The Laplace Transform*, 1st ed. (Princeton, 1946).

[9] E. W. MONTROLL: *Journ. Chem. Phys.*, **10**, 218 (1942); **11**, 481 (1943).

In most cases of interest it is impossible to find more than the first few terms of this series, so the method only suffices for approximate calculations. The Legendre polynomials are particularly useful in this connection since they can be written as

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n} \begin{bmatrix} 2n-2k \\ 2k,\ n-k,\ n-2k \end{bmatrix} x^{n-2k} \tag{27}$$

where $\begin{bmatrix} a \\ b,\ c,\ d \end{bmatrix}$ denotes a trinomial coefficient. However we also have that

$$\sum_{i=1}^{3s} \frac{1}{\pi^3} \int_0^{\pi}\int\int F_j^k(\theta_1, \theta_2, \theta_3)\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{d}\theta_3 = \mu_{2k}\,, \tag{28}$$

where μ_{2k} is the $2k$-th moment of the frequency distribution function. Hence $G(\omega^2)$ can be written as

$$G(\omega^2) = \sum_{n=0}^{\infty} \frac{1}{2n+1} P_n(\omega^2) \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n} \begin{bmatrix} 2n-2k \\ 2k,\ n-k,\ n-2k \end{bmatrix} \mu_{2n-4k}\,, \tag{29}$$

where the μ's can be found from traces of powers of the dynamical matrix.

Still another representation of $\delta(x)$ valid in the finite interval (*) $(-1, 1)$ is

$$\delta(x) = \lim_{n\to\infty} A_n (1-x^2)^n\,, \tag{30}$$

where $A_n = (2n+1)!/[(n!)^2\, 2^{2n+1}]$. This representation leads to the result

$$\begin{cases} G_j(\omega^2) = \dfrac{1}{\pi^3} \dfrac{\omega_L^2}{\omega_L^2 - \omega^2} \lim\limits_{n\to\infty} \dfrac{(2n+1)!}{2^{2n+1}(n!)^2} \displaystyle\int_0^{\pi}\int\int \left[1 - \left(\frac{F_j(\theta_1,\theta_2,\theta_3) - \omega^2}{\omega_L^2 - \omega^2}\right)^2\right]^n \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3\,, \\ \qquad\qquad 0 \leqslant \omega^2 \leqslant \omega_L^2/2\,, \\ = \dfrac{1}{\pi^3} \dfrac{\omega_L^2}{\omega^2} \lim\limits_{n\to\infty} \dfrac{(2n+1)!}{2^{2n-1}(n!)^2} \displaystyle\int_0^{\pi}\int\int \left[1 - \left(\frac{F_j(\theta_1,\theta_2,\theta_3) - \omega^2}{\omega^2}\right)^2\right]^n \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{d}\theta_3\,. \\ \qquad\qquad \omega_L^2/2 \leqslant \omega^2 \leqslant \omega_L^2\,. \end{cases} \tag{31}$$

It is seen that for finite n this leads to a polynomial expansion of $G(\omega^2)$, the coefficients again being found in terms of moments. This method has recently been used in a study of the spectra of disordered lattices ([10]).

(*) We need only restrict our considerations to a finite interval in lattice dynamics problems since it can be shown that the frequency spectrum of such a system must be bounded.

([10]) C. DOMB, A. MARADUDIN, E. MONTROLL and G. WEISS: *Phys. Rev.*, **115**, 8 (1959).

Finally the δ-function formula

$$\int \delta[f(x)]\,\mathrm{d}x = \sum \frac{1}{|f'(x_i)|}\,, \tag{32}$$

where the x_i are simple roots of $f(x) = 0$, is well-known in one dimension. In three dimensions the analogous formula is

$$\iiint \delta[G(\theta_1, \theta_2, \theta_3)]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 = \iint \frac{\mathrm{d}S}{|\mathrm{grad}\, G|}\,, \tag{33}$$

where S is a surface of constant frequency given by $G(\theta_1, \theta_2, \theta_3) = 0$. If we take the element of volume as $\mathrm{d}S\,\mathrm{d}\mu$, where μ is the co-ordinate perpendicular to S, then integrating over μ and noting that the directional derivative along μ is grad G we immediately derive eq. (33) from eq. (12). Thus we have

$$G_j(\omega^2) = \frac{1}{\pi^3} \int\limits_{F_j(\boldsymbol{\theta}) = \omega^2} \frac{\mathrm{d}S}{|\mathrm{grad}\, F_j|}\,. \tag{34}$$

This expression for $G_j(\omega^2)$ has been used by VAN HOVE [11] in a discussion of singularities in frequency spectra.

From our discussion it can be seen that there are as many relations between frequency spectra and dispersion relations as there are representations of the δ-function. We do not expect that the particular cases that we have cited here completely exhaust the range of useful possibilities, but that the variety of possible relations has been suggested by our illustrations.

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[11] L. VAN HOVE: *Phys. Rev.*, **89**, 1189 (1953).

RIASSUNTO (*)

Sono state formulate molte espressioni che mettono in rapporto le relazioni di dispersione con lo spettro di frequenza di un reticolo in vibrazione. È scopo di questo articolo mostrare che queste relazioni si possono tutte derivare da un'unica rappresentazione integrale usando diverse rappresentazioni della funzione δ.

(*) *Traduzione a cura della Redazione.*

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Normalization and Interpretation of Feynman Amplitudes.

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Summary. — A general method is given for normalizing Feynman amplitudes and using them to calculate expectation values. The method is easily applicable to bound states and composite states. Two applications are considered in detail. The first example is the normalization of Bethe-Salpeter amplitudes, where it is found that previously suggested normalization conditions are inadequate. The second example is the renormalization of the electron amplitude for first-order self-energy processes, and here a method is suggested for making the self-energy finite. The self-energy is three times the energy of the bare electron, in this approximation.

1. – Introduction.

The main advantages of Feynman's approach [1] to quantized field theory were the simple physical picture which it provided of the intermediate processes in the interaction of the elementary particles, and the possibility which it seemed to offer of dispensing with the cumbersome traditional techniques of field quantization. The fact that these advantages have not completely matured can be traced to the difficulty which has been experienced in the physical interpretation of the Feynman amplitudes, due mainly to the absence of any clear prescription for the normalization of these amplitudes and the formation of expectation values of quantities of physical interest. While Feynman's approach has been useful (quite apart from its contribution to computational techniques) in suggesting new departures, such as the Salpeter-Bethe equation [2], it has always been necessary to resort to the older methods to provide the necessary justification and physical interpretation.

(1) R. P. Feynman: *Phys. Rev.*, **76**, 749, 769 (1949).
(2) E. E. Salpeter and H. A. Bethe: *Phys. Rev.*, **84**, 1232 (1951).

The present position is that, while the Feynman «rules» are adequate for the normalization and interpretation of amplitudes which involve separate particles in the initial and final states, there is no simple extension of these rules to situations where bound states or composite particles are involved. Progress in this direction has been made through the work of MANDELSTAM [3], NISHIJIMA [4] and ALLCOCK [5], but it cannot be said that the results are intuitively obvious or easy to apply. Also, nothing yet seems to be known about the interpretation amplitudes representing bound states of three or more particles, such as the triton. The present paper has as its object to suggest a general prescription for the interpretation of Feynman amplitudes, which applies to arbitrary systems of independent or composite particles.

Quite apart from problems involving bound states, it will be suggested that this approach may be useful in dealing with questions of renormalization. There is something to be said for the point of view that the difficulties associated with mass and coupling renormalization arise from the fact that elementary («bare») particles are commonly substituted for particles which, in nature, have a composite character. Thus, in nature, an electron cannot be separated from its attendant photon cloud, and the pairs formed by the polarization of the vacuum; it is therefore a composite particle, and if it could be treated as such from the beginning no need for the renormalization techniques would arise. But in fact the methods for dealing consistently with composite particles of this kind are hardly developed. This is clearly a very complex problem, but some indication will be provided below of how at least an approximation solution may be obtained.

The main result on which these developments are based may be stated as follows. To every amplitude u there corresponds a source s, and the combination $p = i(s^\dagger u - u^\dagger s)$ is the probability per unit volume and time that the state represented by the amplitude u will be liberated from the source. This statement provides a normalization for both u and s. The following section is devoted to the demonstration that the consequences are correct and in agreement with known results for independent particles. It will then be shown how the same statement can be applied to systems of particles, and composite particles. One or two examples will be discussed afterwards to show how the method works in detail.

Notation: ∇, or ∇_λ, will represent the four-vector differential operator; ∇^2, or $\nabla^\lambda\nabla_\lambda$, the d'Alembertian operator; $\gamma\cdot\nabla$, or $\gamma^\lambda\nabla_\lambda$ will represent the Dirac operator; $g_{\lambda\mu}$, the metric tensor, has $g_{11} = g_{22} = g_{33} = -1$ and $g_{44} = +1$.

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(3) S. MANDELSTAM: *Proc. Roy. Soc.*, A **233**, 248 (1955).
(4) K. NISHIJIMA: *Prog. Theor. Phys.*, **13**, 305 (1955).
(5) G. R. ALLCOCK: *Phys. Rev.*, **104**, 1799 (1956).

2. – Single particle amplitudes.

In field theory one encounters a variety of relativistic wave equations, for example

$$(\nabla^2 + \mu^2)\varphi = \theta\,, \tag{1a}$$

$$(-i\gamma\cdot\nabla + \mu)\psi = \chi\,, \tag{1b}$$

$$\nabla^2 A_\lambda = J_\lambda\,, \tag{1c}$$

which are all of the general type

$$F(i\nabla)\,u(x) = \sigma(x)\,, \tag{1}$$

where $u(x)$ is the unquantized field variable (scalar, spinor, or vector) and $\sigma(x)$ will be called the field source function.

It is not usual to include in $\sigma(x)$ sources which lie outside the region of observation, and for this reason $u(x)$ is not completely determined by $\sigma(x)$. To overcome this difficulty, the region Ω of observation will be assumed to be defined explicitly, for example by $|x_k| \leqslant L$, $(k = 1, 2, 3)$ and $|x_4| \leqslant T$. A particle created outside this region, or an anti-particle annihilated outside the region, can then be represented by a source function $\sigma_+(x)$ distributed over the boundary of Ω. The field $u_+(x)$ satisfying the equation

$$F(i\nabla)\,u_+(x) = \sigma(x) + \sigma_+(x) = s_+(x) \tag{2}$$

will then differ from $u(x)$ only outside Ω, and will be completely determined by the combined source function $s_+(x)$. In fact, the solution of (2) is

$$u_+(x) = \int K(x - x')\,s_+(x')\,\mathrm{d}^4x'\,, \tag{3}$$

where $K(x)$ is the « causal » Green's function corresponding to the differential operator $F(i\nabla)$, *i.e.*, the Green's function which involves only positive frequencies for large positive values of x_4, and only negative frequencies for large negative values of x_4.

An anti-particle created outside the region Ω, or a particle annihilated outside this region, will be represented by another source function $\sigma_-(x)$ distributed over the boundary. The field $u_-(x)$ defined by

$$\left\{\begin{aligned} &F(i\nabla)\,u_-(x) = \sigma(x) + \sigma_-(x) = s_-(x)\,,\\ &u_-(x) = \int K(x - x')\,s_-(x')\,\mathrm{d}^4x'\,, \end{aligned}\right. \tag{4}$$

will also differ from $u(x)$, and therefore from $u_+(x)$, only outside the region of observation. It should be remarked that $\sigma_+(x)$ involves only positive frequencies, and $\sigma_-(x)$ involves only negative frequencies. An important problem of field theory is to determine $\sigma_-(x)$, where $\sigma_+(x)$ is known, or $\sigma_+(x)$ when $\sigma_-(x)$ is known.

Corresponding to the field variable $u(x)$, there is a conjugate field $u^\dagger(x)$, constructed from but not always equal to the complex conjugate $u^*(x)$. For example, with reference to (1a), (1b) and (1c), the following are conjugates of φ, ψ and A_λ:

$$\varphi^\dagger(x) = \varphi^*(x)\,, \tag{5a}$$

$$\psi^\dagger(x) = \pm\psi^*(x)\gamma_4\,, \tag{5b}$$

$$A^\dagger_\lambda(x) = -A^{\lambda *}(x)\,. \tag{5c}$$

In (5b), the positive or negative sign must be chosen according as $\psi(x)$ involves positive or negative frequencies; if necessary $\psi(x)$ must be resolved into two parts and this rule applied to each part separately. (In practice, it is usually more convenient to construct $\sigma^\dagger_+$ from σ_+, and to deduce the values of $\sigma^\dagger$ and $u^\dagger$ from the conjugate field equation $u^\dagger(x)F(i\nabla^\dagger) = \sigma^\dagger(x) + \sigma^\dagger_+(x)$.)

The usual way of normalizing the field variable is by way of the particle current. The current four-vector corresponding to the three examples enumerated above is

$$j_\lambda = i(\varphi^\dagger\varphi_{,\lambda} - \varphi^\dagger_{,\lambda}\varphi)\,, \tag{6a}$$

$$j_\lambda = \psi^\dagger\gamma_\lambda\psi\,, \tag{6b}$$

$$j_\lambda = i(A^\dagger_\mu A^\mu_{,\lambda} - A^\dagger_{\mu,\lambda}A^\mu)\,, \tag{6c}$$

where $\varphi_{,\lambda}$ means $\partial\varphi/\partial x^\lambda$, etc. The general formula, which will be needed subsequently, has been given by the author [6]: let $G(i\nabla, i\nabla^\dagger)$ be defined by the identity

$$F(i\nabla) - F(i\nabla^\dagger) = -i(\nabla_\lambda - \nabla^\dagger_\lambda)G^\lambda(i\nabla, i\nabla^\dagger) \tag{7}$$

and let $u^\dagger\nabla^\dagger$ be interpreted as $-\nabla u^\dagger$ (so $\nabla^\dagger_\lambda$ means $-\partial/\partial x^\lambda$, acting backwards); then

$$j^\lambda = u^\dagger G^\lambda(i\nabla, i\nabla^\dagger)u\,. \tag{8}$$

If one wishes to describe a situation in which there is just one particle or anti-particle on the space-like surface S, u will be normalized in such a

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[6] H. S. Green: *Proc. Roy. Soc.*, A **197**, 73 (1949), Sect. **4**.

way that

$$\int j^\lambda \, \mathrm{d}S_\lambda = \pm 1 , \tag{9}$$

the positive sign being chosen for a particle, and the negative sign for an anti-particle. If the particle (anti-particle) is a free particle with momentum $\boldsymbol{p}$, which is of course only possible where $\sigma(x)$ in (1) is zero, one sets

$$u = \lambda \exp\left[\pm i(\boldsymbol{p}\cdot\boldsymbol{x} - p_0 x_4)\right], \quad \text{where} \quad p_0 = (\boldsymbol{p}^2 + \mu^2)^{\frac{1}{2}} \quad \text{and for} \quad k = 1, 2, 3,$$

$n_k = Lp_k/\pi$ is an integer; the normalization (9) then gives the well known expressions

$$\varphi(x) = (2p_0 V)^{-\frac{1}{2}} \exp\left[\pm i(\boldsymbol{p}\cdot\boldsymbol{x} - p_0 x_4)\right] ; \tag{10a}$$

$$\psi(x) = (2p_0 V)^{-\frac{1}{2}} \exp\left[\pm i(\boldsymbol{p}\cdot\boldsymbol{x} - p_0 x_4)\right] X_\pm(\boldsymbol{p}) ,$$

$$X_\pm(\boldsymbol{p}) X_\pm(\boldsymbol{p}) = \tfrac{1}{2}\left\{\pm (p_0\gamma_4 - \boldsymbol{p}\cdot\boldsymbol{\gamma}) + \mu\right\}, \tag{10b}$$

$$A_\lambda(x) = (2p_0 V)^{-\frac{1}{2}} \exp\left[\pm i(\boldsymbol{p}\cdot\boldsymbol{x} - p_0 x_4)\right] e_\lambda(\boldsymbol{p}) ,$$

$$e^\lambda(\boldsymbol{p}) e_\lambda(\boldsymbol{p}) = 1 , \qquad p^\lambda e_\lambda(\boldsymbol{p}) = 0 . \tag{10c}$$

Now when the particle is not free, this method of normalization can be applied only with difficulty, and it is convenient instead to apply some condition to the source. It is here that a simple physical interpretation of the field theory is of great assistance. The following is asserted: if $u(x)$ is any field variable, and $s(x)$ the corresponding source, the probability per unit time and per unit volume that a particle will be created by the source, or that an anti-particle will be annihilated by the source, is

$$p(x) = i\left\{u^\dagger(x) s(x) - s^\dagger(x) u(x)\right\}. \tag{11}$$

A negative probability that a particle (anti-particle) will be created (annihilated) is interpreted as a positive probability that a particle (anti-particle) of this kind will be annihilated (created). The probability of finding a particle or anti-particle

$$P_\pm = \pm \int_{-\infty}^{S} p_\pm(x) \, \mathrm{d}^4 x , \tag{12}$$

where $p_\pm(x)$ is the density, constructed from the source function $s_\pm(x)$. The positive sign being chosen for a particle and the negative sign for an anti-particle.

To support the above assertion, it will be shown that it leads to the same normalization as the method already described, where both are applicable. Substituting $F(i\nabla)u(x)$ for $s(x)$ in (11), one finds

$$\left\{\begin{aligned} p(x) &= iu^{\dagger}(x)\left\{F(i\nabla) - F(i\nabla^{\dagger})\right\}u(x) \\ &= u^{\dagger}(\nabla_{\lambda} - \nabla_{\lambda}^{\dagger})G^{\lambda}(i\nabla,\, i\nabla^{\dagger})u \\ &= \nabla_{\lambda} j^{\lambda}(x)\,, \end{aligned}\right. \tag{13}$$

where $j^{\lambda}(x)$ is defined by (9). Thus

$$P_{\pm} = \pm\int j^{\lambda}\,\mathrm{d}S_{\lambda}\,, \tag{14}$$

and the condition (9) is equivalent to the statement that the probability $P_{\pm}$ is 1. It is not necessary to attach the suffix $\pm$ to j^{λ} in (14), because within Ω j_{+}^{λ} and j_{-}^{λ} are the same. From this identity it can be inferred that

$$P_{\pm} = \mp\int_{s}^{\infty} p_{\mp}(x)\,\mathrm{d}^{4}x\,, \tag{15}$$

which is of course necessary for the consistency of the theory.

It is useful to have the normalized sources which create the fields (10a), (10b) and (10c) (apart from a constant factor of modulus 1) within Ω; they are

$$\theta_{\pm}(x) = (2p_0/v)^{\frac{1}{2}} \exp\left[\pm i\boldsymbol{p}\cdot\boldsymbol{x}\right]\delta(x_4 + T)\,, \tag{16}$$

$$\chi_{\pm}(x) = (2\mu v)^{-\frac{1}{2}} \exp\left[\pm i\boldsymbol{p}\cdot\boldsymbol{x}\right]X_{\pm}(p)\,\delta(x_4 + T)\,, \tag{16b}$$

$$J_{\pm}^{\lambda}(x) = (2p_0/v)^{\frac{1}{2}} \exp\left[\pm i\boldsymbol{p}\cdot\boldsymbol{x}\right]e^{\lambda}(p)\,\delta(x_4 + T)\,. \tag{16c}$$

If the corresponding fields are calculated from (3), the probability $P_{\pm}$ defined by (12) is automatically 1.

3. – Interpretation of amplitudes.

The assertion that $p(x)$, defined by (11), represents the rate of production per unit volume, of particles of the particular kind represented by the field variable $u(x)$ suggests the further assertion: if M is the operator representing some measurable quantity associated with the particles produced, the rate of

production of this quantity by the source, per unit volume, will be

$$i\{u^\dagger(x)M^\dagger s(x) - s^\dagger(x)Mu(x)\}.$$

One is thus led to expect that

$$p_\lambda(x) = u^\dagger_{,\lambda}(x)s(x) + s^\dagger(x)u_{,\lambda}(x) \tag{17}$$

will be the rate of production of momentum and energy by the source. Again substituting $F(i\nabla)u(x)$ for $s(x)$, this can be written,

$$p_\lambda(x) = u^\dagger\{-\nabla^\dagger_\lambda F(i\nabla) + F(i\nabla^\dagger)\nabla_\lambda\}u = \tfrac{1}{2}u^\dagger(\nabla_\lambda + \nabla^\dagger_\lambda)\{F(i\nabla^\dagger) - F(i\nabla)\}u + \tag{18}$$
$$+ \tfrac{1}{2}u^\dagger(\nabla_\lambda - \nabla^\dagger_\lambda)\{F(i\nabla^\dagger) + F(i\nabla)\}u = \nabla_\mu T_\lambda{}^\mu,$$

where

$$\left\{\begin{array}{l} T_\lambda{}^\mu = \tfrac{1}{2}iu^\dagger(\nabla_\lambda + \nabla^\dagger_\lambda)G^\mu(i\nabla, i\nabla^\dagger)u - V\delta_\lambda{}^\mu, \\ V(x) = -\tfrac{1}{2}\{s^\dagger(x)u(x) + u^\dagger(x)s(x)\}. \end{array}\right. \tag{19}$$

As the author has pointed out elsewhere [6], $T_\lambda{}^\mu$ is the general formula for the canonical energy-momentum tensor for the field $u(x)$, and

$$\int_{-\infty}^{S} p_\lambda(x) = \int T^\mu_\lambda \, \mathrm{d}S_\mu, \tag{20}$$

is the total energy and momentum associated with the field, on the space-like surface S. The interpretation which as been given to $p_\lambda(x)$, as defined by (17), is thereby confirmed.

Another verification of the interpretation which has been given to $p(x)$ and $p_\lambda(x)$, which is more general though more abstract, may be obtained starting from the observation that $V(x)$, in (19), may be regarded as the action density associated with the creation and annihilation of particles by the source. If one applies the infinitesimal gauge transformation $u \to (1 - i\delta\varepsilon)u$, keeping σ fixed, one finds that

$$V(x) \to V(x) - \tfrac{1}{2}\delta\varepsilon p(x),$$

with $p(x)$ given by (11). Similarly the infinitesimal displacement $u(x) \to u(x+\mathrm{d}x)$ induces the transformation

$$V(x) \to V(x) - \tfrac{1}{2}\delta x^\lambda p_\lambda(x),$$

with $p_\lambda(x)$ given by (17).

A similar consideration determines the rate of generation of angular momentum be the source. The application of an infinitesimal Lorentz transformation to $u(x)$ has the effect

$$u(x^\lambda) \to (1 + iS_{\mu\nu}\delta\omega^{\nu\mu})u(x^\lambda + 2x_k\delta\omega^{k\lambda}) = \tag{21}$$
$$= u(x) - i\delta\omega^{\mu\nu}(S_{\mu\nu} + ix_\mu\nabla_\nu - ix_\nu\nabla_\mu)u(x)\,,$$

where $\delta\omega^{\mu\nu} + \delta\omega^{\mu\nu} = 0$ and $S_{\mu\nu}$ is the tensor operator representing the spin angular momentum. When $s(x)$ is kept fixed, this induces the transformation

$$\begin{cases} V(x) \to V(x) - \frac{1}{2}\delta\omega^{\mu\nu}p_{\mu\nu}(x)\,, \\ p_{\mu\nu}(x) = i(u^\dagger J_{\mu\nu}{}^\dagger s - s^\dagger J_{\mu\nu}u)\,, \\ J_{\mu\nu} = i(x_\mu\nabla_\nu - x_\nu\nabla_\mu) + S_{\mu\nu} \end{cases} \tag{22}$$

in $V(x)$. The tensor $p_{\mu\nu}(x)$ obviously represents the rate of transfer of angular momentum from the source to the particles which it is creating. The invariance of $V(x)$ with respect to co-ordinate displacements and Lorentz transformations, and the symmetry which it exhibits with respect to $u(x)$ and $s(x)$, will ensure that the energy, momentum, and angular momentum gained by the particles are precisely the same as lost by the source. This invariance also ensures that the operators $F(i\nabla)$ and $J_{\lambda\mu}$ will commute, so that, by a calculation similar to (18),

$$\begin{cases} p_{\mu\nu}(x) = \nabla_\lambda M^\lambda_{\mu\nu}\,, \\ M^\lambda_{\mu\nu} = x_\mu T_\nu{}^\lambda - x_\nu T_\mu{}^\lambda + \frac{1}{2}u^\dagger(S^\dagger_{\mu\nu}G^\lambda + G^\lambda S_{\mu\nu})u\,, \end{cases} \tag{23}$$

where $M^\lambda_{\mu\nu}$ may be identified with the angular momentum current tensor of the field.

So far, there has been a derivation in a somewhat generalized and unusual formalism of well known results of quantized field theory. The advantage of this formalism, which has still to appear, is that it is capable of immediate generalization to composite fields. The next section is devoted to this generalization.

4. – Composite field amplitudes.

In a relativistic field theory, such as Feynman's, without second quantization, one is committed to a many-time formalism, with separate space-time co-ordinates, x_a, x_b, etc., for each particle a, b, etc., which one wishes to con-

sider. One has, therefore, an amplitude $u(x_a, x_b, ...)$ for each collection of particles which can be created by the source. If the particles are uncorrelated, $u(x_a, x_b, ...)$ will be a product of amplitudes $u(x_a)u(x_b)$ for the individual particles, suitably symmetrized if necessary to take account of the quantum statistics of identical particles. To every amplitude u there is a corresponding source function s, the two being related by a differential equation of the type

$$F(i\nabla_b, i\nabla_b, ...)\, u(x_a, x_b, ...) = s(x_a, x_b, ...)\,. \tag{24}$$

If the particles interact, the state of the system cannot be characterized by a single amplitude, and one has a set of such equations:

$$F_j u_j = s_j = \sigma_{j+} + \sum_k \sigma_{jk}\,, \qquad (j = 1, 2, ...) \tag{25}$$

in which the sources σ_{jk} are distributed on the boundary of the region of observation, and the σ_{jk} originate on a set of particles represented by the amplitude u_k. Equations of this type can be derived from quantized field theory, as shown, for example, by the work of Caianiello [7]. They can also be formulated directly by semi-intuitive reasoning, based on Feynman's [1] concept of interactions as quantum-stochastic processes, with the space-time co-ordinates as random variables and amplitudes instead of probabilities.

Corresponding to any amplitude u, there is a conjugate amplitude $u^\dagger$, constructed from $u(x_a, x_b, ..)$ as though it were a product $u(x_a)u(x_b)$... of the simple amplitude considered in Section 2. This conjugate amplitude is connected with the conjugate source function $\sigma^\dagger$ by a differential equation of the type

$$u^\dagger(x_a, x_b, ...)\, F(i\nabla_a^\dagger, i\nabla_b^\dagger, ...,) = s^\dagger(x_a, x_b, ...)\,. \tag{26}$$

The equations (25) also have conjugates

$$u_j^\dagger F_j^\dagger = s_j^\dagger = \sigma_j^\dagger + \sum_k \sigma_{jk}^\dagger\,, \tag{27}$$

where $F_j^\dagger = F_j(i\nabla_a^\dagger, i\nabla_b^\dagger, ...)$.

The interpretation of this theory is a fairly obvious generalization of that sketched in the last section. The joint probability, per unit time and per unit volume, that each of a collection of particles $(a, b, ...)$ will be created in the neighbourhood of the points x_a, x_b, ... is $i(u^\dagger s - s^\dagger u)$, and the consequent index of production of an associated physical quantity, represented by the operator M, is $i(u^\dagger M^\dagger s - s^\dagger M u)$.

This bald statement needs to be supplemented by the observation that,

[7] E. R. Caianiello: *Nuovo Cimento*, **11**, 492 (1954).

where one amplitude u_j has as its source σ_{jk} another amplitude u_k, the creation density of the set j by the set k, given by $i(u_j^\dagger \sigma_{jk} - \sigma_{jk}^\dagger u_j)$, must be off-set by the annihilation density of the set k by the set j; the relation

$$\int i(u_j^\dagger \sigma_{jk} - \sigma_{jk}^\dagger u_j)\,\mathrm{d}x_j + \int i(u_k^\dagger \sigma_{kj} - \sigma_{kj}^\dagger u_k)\,\mathrm{d}x_k = 0\,, \tag{28}$$

must therefore hold identically. This serves to determine the relative normalization of the amplitudes u_j and u_k. Another point worth noticing is that the joint creation density

$$p(x_a, x_b, \ldots) = i\{u^\dagger(x_a, x_b, \ldots)\sigma(x_a, x_b, \ldots) - \sigma^\dagger(x_a, x_b, \ldots)u(x_a, x_b, \ldots)\} \tag{20}$$

is not simply the product of the separate creation densities $p(x_a)p(x_b)\ldots$, even where the particles are all different and uncorrelated; however, the difference would be hard to detect observationally, because only integrated expressions involving $p(x_a, x_b, \ldots)$ have direct physical significance.

It will be found helpful to illustrate the above rather abstract discussion with a number of examples. Consider first a single electron, which can exist either as a « bare » particle, with amplitude $u(x)$, or in an excited state with one photon present and amplitude $v(x, y)$ — the space-time co-ordinate y belonging to the photon. Self-energy processes in which two photons are « present » are neglected. The two amplitudes are connected by the equations

$$\begin{cases} (-i\gamma\cdot\nabla + \mu)u(x) = \sigma_+(x) + ie\gamma^\lambda v_\lambda(x, x)\,, \\ \nabla_y^2(-i\gamma\cdot\nabla_x + \mu)v^\lambda(x, y) = \tau_+^\lambda(x, y) - ie\gamma^\lambda\delta(y - x)\,u(x)\,, \end{cases} \tag{30}$$

As

$$\{u^\dagger(x)ie\gamma^\lambda v_\lambda(x, x)\}^\dagger + \int v_\lambda^\dagger(x, y)ie\gamma^\lambda\delta(y - x)u(x)\,\mathrm{d}^4y = 0\,,$$

the condition (28) is already satisfied. The probability for the creation of the electron by the sources $\sigma_+(x)$ nad $\tau_+^\lambda(x, y)$ is

$$P = \int i\{u^\dagger(x)\sigma_+(x) - \sigma_+^\dagger(x)u(x)\}\mathrm{d}^4x + \\ + \iint i\{v^\dagger(x, y)\tau_+(x, y) - \tau_+^\dagger(x, y)v(x, y)\}\mathrm{d}^4x\,\mathrm{d}^4y\,. \tag{31}$$

This must be equated to 1 to yield properly normalized amplitudes for a single electron. The usual practice of quantized field theory amounts to

setting $\tau_+(x, y) = 0$; this, however, has the disadvantage that the normalization factor involves divergent integrals. As explained in Section **6** below, this difficulty can be eliminated by a suitable choice for $\tau_+(x, y)$.

As a second example, consider the formulation of the Salpeter-Bethe ([2]) equation for two fermions, interacting via pseudo-scalar mesons, in the ladder approximation. Strictly speaking, there are two amplitudes here also: one, $u(x_a, x_b)$, to represent a configuration in which no meson is present, and the other, $v(x_a, x_b, y)$ to represent a configuration with the meson « in mid-air ». The equations satisfied by these amplitudes are

$$(32)\qquad \begin{cases} (-\gamma\cdot\nabla_a+\mu)u(x_a, x_b) = \sigma_+(x_a, x_b) + ig\gamma_{5a}v(x_a, x_b, x_a)\,, \\ (\nabla_y^2+\varkappa^2)(-i\gamma\cdot\nabla_b+\mu)v(x_a, x_b, y) = \tau_+(x_a, x_b, y) - ig\gamma_{5b}\delta(y-x_b)u(x_a, x_b)\,. \end{cases}$$

For convenience, Wick's equation ([8]), which has the advantage that it can be solved explicitly ([9]), is stated here in a similar form:

$$(33)\qquad \begin{cases} (\nabla_a^2+\mu^2)u(x_a, x_b) = \sigma_+(x_a, x_b) + iGv(x_a, x_b, x_a)\,, \\ \nabla_y^2(\nabla_b^2+\mu^2)v(x_a, x_b, y) = \tau_+(x_a, x_b, y) - iG\delta(y-x_b)u(x_a, x_b)\,. \end{cases}$$

The normalization of these equations which have so far been suggested ([5]) for bound states ignore the possibility of finding the meson « in the air ». This is perhaps rigorously justifiable in the instantaneous interaction approximation, but appears rather objectionable in principle. It should be said, however, that the amplitude in which the meson is « present » can be eliminated, leading to the equations

$$(34)\qquad (-i\gamma\cdot\nabla_a+\mu)(-i\gamma\cdot\nabla_b+\mu)u(x_a, x_b) = \sigma(x_a, x_b) + \\ + g^2\gamma_{5a}\gamma_{5b}\,\Delta(x_a-x_b)u(x_a, x_b)\,,$$

instead of (32), and

$$(35)\qquad (\nabla_a^2+\mu^2)(\nabla_b^2+\mu^2)u(x_a, x_b) = \sigma(x_a, x_b) + G^2\Delta(x_a-x_b)u(x_a, x_b)$$

instead of (33). (The symbol Δ has been used for the boson Green's functions for zero mass, instead of the usual D, to avoid confusion with another use of D to follow.) The normalization problems which arise in connection with these bound-state equations will be discussed next.

([8]) G. C. Wick: *Phys. Rev.*, **96**, 1124 (1954).
([9]) H. S. Green: *Nuovo Cimento*, **5**, 866 (1957).

5. – Normalization of bound states.

The pairs of equations (32) and (33) will both be written in the abbreviated form

$$\begin{cases} D_a u = \sigma_+ + iGv_v\,, \\ D_y D_b v = \tau_+ - iG\delta u\,, \end{cases} \tag{36}$$

and the notation

$$\begin{cases} D_a - D_a^\dagger = -i(\nabla_a - \nabla_a^\dagger)_\lambda G_a^\lambda \\ D_b - D_b^\dagger = -i(\nabla_b - \nabla_b^\dagger)_\lambda G_b^\lambda \\ D_y - D_y^\dagger = -i(\nabla_y - \nabla_y^\dagger)_\lambda G_y^\lambda \end{cases} \tag{37}$$

will be used. The creation density for the amplitude u is

$$C_u = i(u^\dagger D_a u - u^\dagger D_a^\dagger u) = \nabla_{a\lambda}(u^\dagger G_a^\lambda u)\,, \tag{38}$$

and for the amplitude v,

$$\begin{aligned} C_v &= i(v^\dagger D_y D_b v - v^\dagger D_b^\dagger D_y^\dagger v) = \\ &= \tfrac{1}{2}\nabla_{b\lambda}\{v^\dagger(G_b^\lambda D_y + D_y^\dagger G_b^\lambda)v\} + \tfrac{1}{2}\nabla_{y\lambda}\{v^\dagger(G_y^\lambda D_b + D_b^\dagger G_y^\lambda)v\}\,. \end{aligned} \tag{39}$$

Setting $x = \frac{1}{2}(x_a + x_b)$, $\xi = x_a - x_b = 2(x_a - x) = 2(x - x_b)$ and $\eta = y - x$, one has

$$\nabla_a = \tfrac{1}{2}(\nabla_x + 2\nabla_\xi - \nabla_\eta)\,, \qquad \nabla_b = \tfrac{1}{2}(\nabla_x - 2\nabla_\xi - \nabla_\eta)\,, \qquad \nabla_y = \nabla_\eta\,, \tag{40}$$

so that if C_u is integrated over all values of ξ, and C_v over all values of ξ and η, one finds for the creation density of the mean centre x

$$\begin{aligned} C &= \int C_u \,\mathrm{d}^4\xi + \iint C_v \,\mathrm{d}^4\xi\,\mathrm{d}^4 y = \\ &= \nabla_{x\lambda}\left\{\frac{1}{2}\int u^\dagger G_a^\lambda u \,\mathrm{d}^4\xi + \frac{1}{4}\int v^\dagger(G_b^\lambda D_y + D_y^\dagger G_b^\lambda)\, v \,\mathrm{d}^4\xi\,\mathrm{d}^4\eta\right\}. \end{aligned} \tag{41}$$

The amplitudes u and v will be correctly normalized, provided

$$\int\left\{\frac{1}{2}\int u^\dagger G_a^\lambda u \,\mathrm{d}^4\xi + \frac{1}{4}\int v^\dagger(G_b^\lambda D_y + D_y^\dagger G_b^\lambda)\, v \,\mathrm{d}^4\xi\,\mathrm{d}^4\eta\right\}\mathrm{d}S_\lambda = 1\,, \tag{42}$$

where S is any space-like surface intersecting the region Ω within which the bound state exists.

This normalization is *entirely different* from that suggested by the work of ALLCOCK ([5]), BISWAS ([10]) and ALLCOCK and HOOTON ([11]), which even requires u to have different dimensions. The difference can be attributed intuitively to the fact that these authors did not consider the possibility of finding the intermediate boson «in the air». As both of the amplitudes u and v represent virtual states, if the particles a and b are bound, there is no satisfactory basis for attributing more importance to one than to the other; unless, perhaps, the instantaneous interaction approximation is adopted, when the intermediate boson has an arbitrarily short existence. One is therefore obliged to prefer the normalization of (42) to that which has been used hitherto. Yet it must be freely admitted that the method of normalization used may not affect the calculated expectation value of a particular physical quantity; in fact, it will not if the latter is a constant of the motion. One will not therefore easily find an experimental test to distinguish between the two methods of normalization; photo-meson production in deuterium might provide a counter-example, but even here one could no doubt calculate the cross-section correctly by suitably processing an amplitude which has been normalized by Allcock's method.

It will perhaps increase confidence in the method of normalization suggested here if it is applied to obtain Allcock's result. This may, in fact be done very easily, proceeding from the equation (34) or (35), instead of (32) or (33), *i.e.*, from

$$D_a D_b u = \sigma + G^2 \Delta u \, .$$

The associated creation density is

$$(43) \qquad \begin{cases} C'_u = i(u^\dagger D_a D_b u - u^\dagger D_b^\dagger D_a^\dagger u) \\ \quad = \tfrac{1}{2}\nabla\{u^\dagger(G_a^\lambda D_b + D_b^\dagger G_a^\lambda)u\} + \tfrac{1}{2}\nabla_{b\lambda}\{u^\dagger(G_b^\lambda D_a + D_a^\dagger G_b^\lambda u\} \end{cases}$$

and this gives normalization

$$(44) \qquad \int_{-\infty}^{S} C'_u \, \mathrm{d}^4\xi \, \mathrm{d}^4x = \frac{1}{2}\iint u'^\dagger (G_a^\lambda D_b + D_b^\dagger G_a^\lambda) u \, \mathrm{d}^4\xi \, \mathrm{d}S_\lambda = 1 \, .$$

As the notation here differs somewhat from that used by earlier authors, the same result is derived briefly by Biswas's method ([10]) (which is equivalent to ALLCOCK and HOOTON's ([11])) in an Appendix to this paper.

([10]) S. N. BISWAS: *Nuovo Cimento*, **7**, 877 (1958), Appendix II.

([11]) G. R. ALLCOCK and D. J. HOOTON: *Nuovo Cimento*, **8**, 590 (1958).

6. – Normalization with self-energy processes.

Next the example introduced in Section **4** will be discussed in more detail. The equations (30) may be written briefly in the form

$$\left\{\begin{aligned} D_u &= \sigma_+ + Ev_0\,, \\ D_y D_v &= \tau_+ - E\delta_u\,, \end{aligned}\right. \tag{45}$$

where $v_0 = v(x, x)$. The creation densities are

$$\begin{aligned} C_u &= iu^\dagger(D - D^\dagger)u\,, \\ &= \nabla_\lambda(u^\dagger G^\lambda u)\,, \end{aligned}$$

for the bare electron, and

$$\begin{aligned} C_v = iv^\dagger(D_y D - D^\dagger D_y^\dagger)v = \\ = \tfrac{1}{2}\nabla_\lambda\{v^\dagger(G^\lambda D_y + D_y^\dagger G^\lambda v\} + \nabla_{y\lambda}\{v^\dagger(G_y^\lambda D + D^\dagger G_y^\lambda)v\} \end{aligned} \tag{47}$$

for the excited state. On integrating over all values of y, the second term in (47) disappeares, and the electron particle current is found to be

$$j^\lambda = u^\dagger G^\lambda u + \frac{1}{2}\int v^\dagger(G^\lambda D_y + D_y^\dagger G^\lambda)v\,\mathrm{d}^4y\,. \tag{48}$$

This should be normalized in such a way that $\int j^\lambda \mathrm{d}S_\lambda = 1$.

By using the current for normalization one eliminates the need to exhibit the source functions σ_+ and τ_+ explicitly. But of course u and v depend on σ_+ and τ_+, so that their absence from the normalization formula is only apparent. The usual renormalization procedure is to assume a « bare » source, *i.e.*, to take $\tau_+ = 0$, when integration of the second of equations (45) gives

$$v(x, y) = \int \Delta(y - z)\,S(x - z)\,Eu(z)\,\mathrm{d}^4z\,, \tag{49}$$

where Δ and S are the causal Green's functions corresponding to D_y and D respectively. When one sets $y = x$ to form v_0, the coincidence of the singularities of $\Delta(x - z)$ and $S(x - z)$ makes the integral divergent. In spite of this divergence, the normalization can be carried out formally by the method described, and results in renormalized amplitudes u and v. The mathematically

unsatisfactory nature of this procedure is apparent from the fact that the bare mass μ (expressed in terms of the physical mass) and the source function σ_- both involve a meaningless divergence. It is interesting to see how this eventuality is affected by introducing an excited source τ_-, as well as the bare source σ_-. The facility with which this can be done is the main advantage of the present formalism.

There is, however, some difficulty in deciding what τ ought to be. The wrong choice would mean that one was describing an electron in interaction with some external field, just outside the region Ω, instead of a free electron, and would generate states including, for example, those which arise virtually in Compton scattering. There is some advantage in postponing the consideration of this difficulty and writing

$$\tau_+ = D_y D v_a + E\,\delta u_a\,, \tag{50}$$

where $u_a(x)$ and $v_a(x, y)$ are left arbitrary, except that of course u_a and $D_y D v_a$ must vanish if x and y lie within the region Ω of observation. When (50) is used, (49) is replaced by

$$v = v_a + \int \Delta S E(u - u_a)\,\mathrm{d}^4 z\,, \tag{51}$$

if the co-ordinate variables are omitted for the sake of brevity.

At first sight (51) looks little better than (49), from the point of view of convergence. It is possible to choose u_a equal to u outside the region Ω, so that no divergence arises there, and in this respect the use of an excited source has been advantageous. On the other hand u_a has to vanish in Ω, and there are still coincident singularities there in general. But it will be shown that these singularities disappear, when a suitable condition is imposed on u.

Write

$$S(x - z) = (-i\gamma\cdot\nabla_z + \mu)\,\Delta_\mu(x - z)\,, \tag{52}$$

or, more briefly, $S = D_z \Delta_\mu$. Then

$$E\Delta S E u = \tfrac{1}{2} E\left\{\Delta(D_z\Delta_\mu) - (D_z\Delta)\Delta_\mu\right\} E u + \tfrac{1}{2} E\left\{\Delta(D_z\Delta_\mu) + (D_z\Delta)\Delta_\mu\right\} E u\,. \tag{53}$$

In the first term on the right-hand side, the coincident singularities cancel when $y \to x$, and there will be no divergence when it is integrated with respect to z in Ω. If one uses

$$E D_z E = -e^2\gamma^\lambda D_z \gamma_\lambda = -2e^2(i\gamma\cdot\nabla_z + 2\mu)\,, \tag{54}$$

the last term becomes

$$\tfrac{1}{2}E\{\Delta(D_z\Delta_\mu)+(D_z\Delta)\Delta_\mu\}\,Eu=-ie^2\gamma\cdot\nabla_z(\Delta\Delta_\mu u)-e^2\Delta\Delta_\mu D_z'u\,, \tag{55}$$

where

$$D_z'=-i\gamma\cdot\nabla_z+4\mu\,. \tag{56}$$

The term $-ie^2\gamma\cdot\nabla_z(\Delta\Delta_\mu u)$ reduces on integration to a surface integral on Ω, where it can be compensated by an equal and opposite contribution from u_a; this term can therefore be disregarded. The remaining term $-e^2\Delta\Delta_\mu D_z'u$ will will vanish, provided

$$(-i\gamma\cdot\nabla+4\mu)u(x)=0\,, \tag{57}$$

within Ω, *i.e.*, provided the physical mass is four times the bare mass, in this second-order approximation. If this condition is satisfied, all divergences can be made to disappear by the suitable choice of u_a. The other function, v_a, contributing to τ_+ is still undetermined, and can be used to secure the equivalence of (57) and the first of equations (45), *i.e.*, to make

$$Ev_0=-3\mu u \tag{58}$$

within Ω. The equation (51) gives

$$Ev=Ev_a+\frac{1}{2}\int E\{\Delta(D_z\Delta_\mu)-(D_z\Delta)\Delta_\mu\}Eu\,\mathrm{d}^4z\,, \tag{59}$$

and the elimination of c between (58) and (49) gives the desired condition on v_a.

It is interesting that the mass renormalization constant can be made finite by a proper choice of source function. This has been proved here only for second order self-energy processes of a free electron but there is no obvious difficulty in extending the method to higher order self-energy processes, and it is known ([12]) that in interactions with photons self-energy and vertex-type divergences cancel. What has not been shown is that the polarization of the vacuum can be treated in a similar way, and the author does not know at present whether this is possible (*). If it were possible, the chances of reformulating quantum electrodynamics and other renormalizable field theories in a consistent way would appear much brighter.

([12]) J. C. Ward: *Phys. Rev.*, **78**, 182 (1950).

(*) *Note added in proof.* – It is, at least for free photons.

Appendix

Consider the equation

$$\left\{\begin{aligned} &D_a D_b U = \Sigma_+ + G^2\,\Delta U, \qquad \Delta = \Delta(x_a - x_b)\,, \\ &\Sigma_+(x_a, x_b) = \delta(x_a - y_a)\,\delta(x_b - y_b)\,. \end{aligned}\right. \tag{60}$$

This is the same as the equation considered in Sect. 5, except that a special choice has been made for the source function which makes U a function of y_a and y_b as well as x_a and x_b. However, the solution of $D_a D_b u = \sigma_+ + G^2 \Delta u$ can be expressed as

$$u(x_a, x_b) = \int U(x_a, x_b; y_a, y_b)\,\sigma_+(y_a, y_b)\,\mathrm{d}^4 y_a\,\mathrm{d}^4 y_b\,. \tag{61}$$

The differential equation (60) may be replaced by a corresponding integral equation

$$\begin{aligned} U(x_a, x_b; y_a, y_b) = {} & U_+(x_a, x_b; y_a, y_b) + \\ & + G^2 \int U_+(x_a, x_b, z_a, z_b)\,\Delta(z_a - z_b)\,U(z_a, z_b; y_a, y_b)\,\mathrm{d}z_a\,\mathrm{d}z_b\,, \end{aligned} \tag{62}$$

which can be written briefly $U = U_+ + G^2 U_+ \Delta U$. A similar equation holds in the momentum representation, and it is this which is used for normalization purposes. Explicitly this is

$$U(p, q) = U_+(p, q) + G^2 \iint U_+(p, r)\,\Delta(r, s)\,U(s, q)\,\mathrm{d}^4 r\,\mathrm{d}^4 s\,, \tag{63}$$

where $U(p, q)$ is defined by

$$\begin{aligned} \iiiint U(x_a, x_b; y_a, y_b)\,\exp\left[\,i(q_a y_a + q_b y_b - p_a x_a - p_b r_b)\,\right]\mathrm{d}^4 x_a\,\mathrm{d}^4 x_b\,\mathrm{d}^4 y_a\,\mathrm{d}^4 y_b = \\ = U(\tfrac{1}{2}p_a - \tfrac{1}{2}p_b,\ \tfrac{1}{2}q_a - \tfrac{1}{2}q_b)\,(2\pi)^4\,\delta(p_a + p_b - q_a - q_b)\,. \end{aligned} \tag{64}$$

Although U is shown as a function of $p = \frac{1}{2}(p_a - p_b)$ and $q = \frac{1}{2}(q_a - q_b)$, it depends in fact also on the resultant energy-momentum $P = p_a + p_b$. If there is a bound state with energy E, U will have a pole for $P^2 = E^2$, and can therefore be written in the form

$$U = (P^2 - E^2)^{-1} W\,. \tag{65}$$

Assuming the energy level is non-degenerate, W will reduce to

$$W = u(p)\,u^{\dagger}(q)\,, \qquad \text{for } P^2 = E^2. \tag{66}$$

Now, from (63) it follows that

$$W = (P^2 - E^2)\, U_+ + G^2 U_+ \Delta W = \tag{67}$$

$$= (P^2 - E^2)\, U_+ + G^2 W \Delta U_+ ,$$

for arbitrary values of P. Differentiating the first line of this equation with respect to P_λ, multiplying from the left by $W\Delta$, and using the second line, one finds, for $P^2 = E^2$,

$$2P_\lambda W \Delta U_+ + G^2 W \Delta \left(\frac{\partial U_+}{\partial P^\lambda}\right) \Delta W = 0 ,$$

i.e.,

$$2P_\lambda + G^2 u^\dagger \Delta \left(\frac{\partial U^+}{\partial P^\lambda}\right) \Delta u = 0 , \tag{68}$$

which can also be written

$$2P_\lambda = u^\dagger \left\{\frac{\partial}{\partial P^\lambda} (D_a D_b)\right\} u , \tag{69}$$

since $D_a D_b$ is the reciprocal of U_+, and $G^2 U - \Delta u = u$ within Ω. It is readily seen that this condition is equivalent to (44), when the latter is converted to the momentum representation.

RIASSUNTO (*)

Si espone un metodo generale per normalizzare le ampiezze di Feynman e per usarle per calcolare i valori di aspettazione. Il metodo si può applicare facilmente a stati legati e a stati composti. Si studiano dettagliatamente due applicazioni. Il primo esempio è la normalizzazione delle ampiezze di Bethe-Salpeter, in cui si trova che le condizioni di normalizzazione precedentemente suggerite sono inadeguate. Il secondo esempio è la rinormalizzazione delle ampiezze elettroniche per processi autoenergetici del primo ordine. L'autoenergia è il triplo dell'energia dell'elettrone nudo, in questa approssimazione.

(*) *Traduzione a cura della Redazione.*

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A New Derivation of the Statistical Theory of Particle Production with Numerical Results for p-p Collisions at 25 GeV.

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Summary. — A new derivation of the statistical model for particle production is given, starting from S-matrix theory. Though the final formulae are essentially those used already in other recent publications, this derivation shows up clearly where the heuristic arguments come in and which are the weak points. One also sees that, contrary to a widespread opinion, the angular isotropy in the centre-of-mass system is neither a consequence, nor a pre-supposition of the statistical theory; the results of such a theory merely refer to averages over all angles in the CM-system. A critical discussion shows, however, that even the present form of the theory applies only to central collisions (in a wide sense) and should be extended to non-central ones, if one wishes very detailed data. Arguments are given which show that nevertheless the central collision theory may give agreement with experiments for many interesting quantities (as experience shows for 2.75 GeV and 6.2 GeV). A more detailed differentiation of the notion of inelasticity is proposed. Spectra and mean production numbers for 25 GeV p-p collisions are given in form of curves.

1. – Introduction.

In this paper it is tried to predict the details of particle production in p-p collisions at 25 GeV primary kinetic energy of the incoming proton. The results will be valid (if at all) only for p-p collisions, they will be similar for p-π collisions and may be considerably different for nucleon-nucleus collisions. The tables and graphs apply therefore to experiments in which a liquid hydrogen target is used. Furthermore, the theory strictly speaking supposes perfect head-on collisions, but it should in fact be valid also for other ones, as long as the inelasticity becomes not too small. The total cross-section for the collisions considered cannot be calculated, instead we express the numbers of

particles produced by saying « per event », which means that the number of events must be still found from other considerations, as on beam intensity, target density and the cross-section. Since those events, for which the theory is hoped to give good predictions, should have high inelasticity, one may assume that the cross-section for them is somewhat lower than the geometric one, probably one quarter to one times the latter.

Everything is expressed in the centre-of-mass system (CMS). The units are $\hbar = c = M_{\text{Nucl}} = 1$.

Nothing can be said concerning the angular distribution in CMS and also nothing is known about the correlation between angle and energy. The energy spectra given below mean, in fact, an average over all angles in CMS, *but do in no way suppose or imply isotropy*. The underlying theory is basically the old low energy (not thermodynamic) Fermi-theory [1] with some essential additions and carried through rigorously. We shall derive the formalism in the next paragraph and discuss it in detail.

In the last few years, the Fermi model has met with more success after having been discredited for quite a time. Its failing was partly due to its incompleteness, partly to the fact that inadequate approximations were used in calculations and sometimes essential parts neglected for convenience.

The merit of removing these different sources of disagreement between theory and experiment is shared between many authors. The most essential points were:

i) Inclusion of final state interactions [2-7] as: Resonant states, decay of particles, final state annihilation.

ii) Inclusion of isospin conservation by giving tables of coefficients for direct use [8,9].

iii) Exact values for the phase space integrals [10-12].

iv) Allowing for different types of interaction [13].

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(1) E. Fermi: *Prog. Theor. Phys.*, **5**, 570 (1950).

(2) S. Z. Belen'kij V. M. Maksimenko, A. I. Nikišov and I. L. Rozental': *Uspechii Fiz. Nauk*, **62**, 1 (1957). A German translation appeared in *Fortschr. d. Phys.*, **6**, 524 (1958).

(3) S. Z. Belenij: *Nucl. Phys.*, **2**, 259 (1956-57).

(4) J. Kowacs: *Phys. Rev.*, **101**, 397 (1956).

(5) S. J. Lindenbaum and R. M. Sternheimer: *Phys. Rev.*, **105**, 1874 (1957).

(6) F. Cerulus and R. Hagedorn: *CERN* 59-3 (yellow CERN-publication).

(7) R. Hagedorn: *Nuovo Cimento*, **15**, 246 (1960).

(8) Y. Yeivin and A. de-Shalit: *Nuovo Cimento*, **1**, 1147 (1955).

(9) V. S. Barašenkov and B. M. Barbašev: *Suppl. Nuovo Cimento*, **7**, 19 (1957).

(10) M. M. Block: *Phys. Rev.*, **101**, 796 (1956).

(11) G. E. A. Fialho: *Phys. Rev.*, **105**, 328 (1957).

(12) F. Cerulus and R. Hagedorn: *Suppl. Nuovo Cimento*, **9**, 646, 659 (1958).

(13) V. S. Barašenkov, B. M. Barbašev, E. G. Bubelov and V. M. Maksimenko: *Nuclear Phys.*, **5**, 17 (1958).

The Fermi model, thus polished up, showed in fact good agreement with experiments at 2.75 GeV ([2,6]) and 6.2 GeV ([7]) primary energy and, with inclusion of a π-π interaction with a resonant state, was also able to give a detailed description of nucleon-antinucleon annihilation, which fits almost too well to the experiments ([14]).

Some of the above listed cases were not too difficult to compute, since the total energy was low and consequently only a few final states were in competition. However, since each final state has to be considered separately and since their total number increases rapidly with energy, the temptation has been great to consider only a few end-states, assume them to be typical and omit the others. This leads to large errors. For 25 GeV one finds a mean pion number per collision of

5.35 if only $2\mathcal{N}+n\pi$ end-states are considered

4.23 if also end-states with $\mathcal{N}$-$\mathcal{N}$ pairs are included

3.37 if all end-states with non-negligeable weight are counted (*).

Even if one is only interested in pions, one is thus forced to consider all contributing end-states, since those which produce only one or two pions together with hyperons, anti-nucleons, anti-hyperons, K-mesons etc., may have a considerable probability and thus push down the mean pion number.

In some comparisons of experimental data with a (very simplified and sometimes incorrect) Fermi type calculation, this latter fact also has been disregarded and very bad agreement was found. The removal of this inconsistency presents, however, its own difficulties. The number of contributing end-states (labelled by particle-numbers and kinds as *e.g.* $\mathcal{N}+\mathcal{N}+2\pi$, $\mathcal{N}+\Sigma+K+\pi$ etc.) is roughly

$$10 \quad \text{for} \quad 2.75 \text{ GeV},$$

$$50 \quad \text{for} \quad 6.2 \text{ GeV},$$

$$600 \quad \text{for} \quad 25 \text{ GeV}.$$

Since for each final state and each kind of particle in it, an energy spectrum can be calculated (apart from its total probability) and each spectrum may consist of, say, 30 numbers, one sees that the amount of numerical material becomes discouragingly large. It was therefore necessary to develop a programme for doing this automatically on an electronic computer. For energies higher than 3 GeV this seems to be an indispensable tool, which (with the

([14]) F. CERULUS: *Nuovo Cimento*, **14**, 827 (1959).

(*) These figures are not corrected for final state annihilation and should consequently not be used in other contexts.

present speeds of computers) breaks down again for energies higher than, say, 50 GeV. Above that energy the thermodynamic and hydrodynamic method [15] developed for the very high energies must be used. It should be interesting to investigate how the thermo-hydrodynamic extrapolations down to 25—50 GeV fit to the present calculations.

The above mentioned programme [16,17] has been used extensively at CERN for calculations at 6.2 GeV [7], annihilation [14], 10 GeV (not published, still incomplete) and the present 25 GeV case.

Even if in the last few years the Fermi theory was developed to a stage where many experimental facts could be reproduced, this does not mean that it could remain as it is today. In particular, three related questions, namely the non-central collisions, the conservation of angular momentum and the angular distribution are not yet answered and so far there is not even a way known how to attack them without using very naive models.

2. – The model.

2·1. *Derivation of an exact formula from S-matrix theory.*

a) Probability for a set F of final states. – We shall give here a derivation of the «statistical model» which shows that it can be derived to a wide extent from S-matrix theory. Since a rigorous S-matrix theory (which allows to calculate the matrix elements) does hardly exist for 2-particle systems, it should be clear that in the present few- or many-particle problem, some heuristic arguments must be used, leading to a formalism which should be called a «model», rather than a theory. The model has been described frequently in the literature [1,2,6,18] and our present derivation leads to the same results, but gives perhaps some better understanding.

Assume, we start from a definite initial state i and wish to calculate the transition probability to a certain set F of possible end-states f, then from the definition of the S-matrix, this probability is:

$$P(F) = \sum_{f \in F} |\langle f|S|i\rangle|^2 . \tag{1}$$

If the initial state is not pure, one has to average over its components.

(15) E. L. Feinberg: *High-Energy Conference, Kiev* (1959).
(16) R. Hagedorn: *CERN* 59-25 (yellow CERN-publications).
(17) R. Hagedorn: *Calculating high energy particle production according to statistical theories on an electronic computer*, internal CERN report 1959. Available on request.
(18) R. Milburn: *Rev. Mod. Phys.*, **27**, 1 (1955).

Let us specify the final states by the number of particles, their masses and three-momenta, their spins (not explicitely written) and their isospins (T = total isospin, ϑ = third component)

$$P(F) = \sum_{\widetilde{F}} \int \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_n \, |\langle T_1 \vartheta_1 \dots T_n \vartheta_n, \, \boldsymbol{p}_1 \dots \boldsymbol{p}_n, \, m_1 \dots m_n | S | i \rangle|^2 . \tag{2}$$

Here F means a sum over isospin variables and integration over momenta; both restricted according to which set of final states is considered.

b) Reduction of the matrix elements. – We may reduce the matrix element by imposing some conditions on the interation and separating off the corresponding conservation laws. We demand:

i) Energy-momentum conservation,

ii) Isospin conservation,

iii) Baryon number conservation,

iv) Strangeness conservation.

i) and ii) will be made explicit, whereas iii) and iv) will be obeyed by the convention that in writing down final states, one never writes states which would violate them.

Having once chosen the momenta for the description of the final states, one cannot make explicit the conservation of angular momentum (and vice versa), which certainly is one of the most serious disadvantages. It is, of course, hidden in the matrix element, but so far nobody succeeded in getting it out. Since these observables do not commute, this seems to be indeed difficult.

c) Isospin and energy-momentum conservation. – For the factorization, with respect to the conservation law ii), we have to expand the final state into isospin eigenfunctions. The final state is in fact a product of single particle isospin functions:

$$|T_1 \vartheta_1\rangle \, |T_2 \vartheta_2\rangle \dots |T_n \vartheta_n\rangle$$

and as such, not yet an eigenfunction of the total isospin T and its third component ϑ. We must therefore find its projection onto such a pure state using the well known formalism of coupling isospins. For the coupling of two such particles to a resultant $T = t_2$, $\vartheta = \tau_2$ we have:

$$|T_1 \vartheta_1\rangle \, |T_2 \vartheta_2\rangle = \sum_{t_2 = |T_1 - T_2|}^{T_1 + T_2} \sum_{\tau_2 = -t_2}^{t_2} C_{T_1 T_2}(t_2 \tau_2 | \vartheta_1 \vartheta_2) \cdot |t_2 \tau_2\rangle_{T_1 T_2} .$$

Proceeding that way by coupling the third particle to this function, then the fourth etc., we find (with $t_1 = T_1$):

$$(3) \qquad \prod_{i=1}^{n} |\, T_i \vartheta_i \rangle = \Big[\prod_{i=2}^{n} \sum_{t_i = |t_{i-1} - T_i|}^{t_{i-1}+T_i} \sum_{\tau_i = -t_i}^{t_i} C_{t_{i-1} T_i}(t_i \tau_i \,|\, \tau_{i-1} \vartheta_i) \Big] \, |\, t_n \tau_n \rangle_{T_1 T_2 \ldots T_n} .$$

Isospin conservation implies that

$$\langle t_n \tau_n \,|\, S \,|\, T \vartheta \rangle = 0 \qquad \text{unless } t_n = T,\ \tau_n = \vartheta.$$

If, with these values for t_n and τ_n we denote the square bracket in (3) by

$$C_T(T_1 \ldots T_n,\ \vartheta_1 \ldots \vartheta_n)$$

then we find for the square matrix element

$$(4) \qquad |\langle T_1 \vartheta_1 \ldots T_n \vartheta_n,\ \boldsymbol{p}_1 \ldots \boldsymbol{p}_n,\ m_1 \ldots m_n \,|\, S \,|\, i \rangle|^2 =$$
$$= |\, C_T(T_1 \ldots T_n,\ \vartheta_1 \ldots \vartheta_n)|^2 \cdot |\langle \boldsymbol{p}_1 \ldots \boldsymbol{p}_n,\ m_1 \ldots m_n \,|\, S' \,|\, i \rangle_{T_1 \ldots T_n}|^2 .$$

Energy-momentum conservation can be accounted for by a factor $\delta(E - E_f)\, \delta(\boldsymbol{P} - \boldsymbol{P}_f)$.

d) The number of final states; angular momentum. – Furthermore, we know that what we do in fact is counting in a weighted sum (the weight factor is the square matrix element) the final states contained in the set F. From elementary quantum mechanics, it is well known that the number of states of n free particles with total energy between E and $E + \Delta E$ in a box of volume V is

$$(5) \qquad \Delta E \cdot \frac{V^{n-1}}{(2\pi\hbar)^{3n-3}} \int \mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_n \, \delta\Big(E - \sum_j \varepsilon_j\Big)\, \delta\Big(\boldsymbol{P} - \sum_j \boldsymbol{p}_j\Big) ,$$

if the total momentum is prescribed. The exponent $n - 1$ comes from the momentum δ-function.

So we expect a factor $[V/(2\pi)^3]^{n-1}$ to be present in the matrix element squared (remember $\hbar = 1$). There are still two corrections to be made:

i) If there were not the sum and the integral over F but just the $\mathrm{d}\boldsymbol{p}_1 \ldots \mathrm{d}\boldsymbol{p}_n$ and the $\vartheta_1 \ldots \vartheta_n$ fixed, then that would mean that we ask for a definitive distribution of momenta and charges by which the particles are then labelled. If, however, we integrate over momenta, then all possible permutations of momenta contribute. Since permutations among equal particles

should not give a new state, we obviously have to compensate by a further factor $1/(N_1!\,N_2!\,\dots\,N_j!)$) where $N_1 \dots N_j$ are the numbers of equal particles. Having introduced isospin explicitly, all members of one isospin multiplet have to be considered as equal.

ii) That would be right now, if the particles spins had been introduced explicitly also. Since that is not the case, we must account for the different spin states by a factor $2s+1$ for each spin s. This gives $\prod_j (2s_j+1)^{\sigma_j}$ if there are $\dots \sigma_j \dots$ particles with spin $\dots s_j \dots$

One might ask here, why the spins are not treated the same way as the isospins. The answer is that there is still the orbital angular momentum of each particle j (which is not defined, as it does not commute with $\boldsymbol{p}_j$) and these orbital angular momenta had to be coupled to the spins to give the total angular momentum, which also is not defined (*). It might be not unreasonable to say that momenta and angular momenta are both unsharp, but still sharp enough to be both conserved explicitly by δ-like functions. In that case, one had to do for the angular momenta the same as for isospin. This whole question, though of practical importance, seems to be still not understood. At least angular momentum conservation does not play an important quantitative role. In the very similar case of nuclear evaporation this has been shown some years ago ([19]).

So at present it seems most reasonable to disregard angular momentum at all, and simply add the above factor for the statistical weight of spins.

Writing all aforementioned factors explicitly, formula (2) changes into

$$(6)\qquad P(F) = \left|\frac{V}{(2\pi)^3}\right|^{n-1} \cdot \frac{\prod (2s_j+1)^{\sigma_j}}{\prod \bar{N}_i!} \sum_{\vartheta_j \in F} |\, C_T(T_1 \dots T_n,\ \vartheta_1 \dots \vartheta_n)\,|^2 \cdot$$

$$\cdot \int_F |\, \langle \boldsymbol{p}_1 \dots \boldsymbol{p}_n,\ m_1 \dots m_n\, S\, |\, i \rangle T_1 \dots T_n\,|^2 \cdot \delta(E - E_f)\ \delta(\boldsymbol{P} - \boldsymbol{P}_f)\, \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_n\,,$$

(we still write S for convenience, though it has a different meaning now).

e) The set of final states considered. – The set F of final states needs perhaps some comments. If we are interested in the charge distribution, say by asking for the probability of finding a certain final state like $\mathcal{T}^- - \mathcal{T}^0 + 2\pi^+ + \pi^-$, then the ϑ_j are fixed and there is no sum over them. The expressions $C_T(T_1 \dots T_n,\ \vartheta_1 \dots \vartheta_n)^2$, which then appear explicitly are complicated, as one sees from their definition in formula (3). A table of them is available ([20]). For small particle numbers such an analysis has been carried

(*) See the remark following (8) on the next page.

([19]) R. HAGEDORN: *Zeits. f. Naturf.*, **9***a*, 259 (1954).

([20]) F. CERULUS: *Nuovo Cimento*, in press.

through already ([3,6,14]) or is in progress ([21]). At higher particle numbers, as for instance in the present case (25 GeV primary energy), the amount of calculation necessary for charge analysis is not justified, since the simple rule that each member of an isospin multiplet has about the same frequency, is already a good approximation. Consequently, what concerns the charges, we take F to include all possible states, which amounts to sum over all ϑ_j from $-T_j$ to $+T_j$. This sum is in fact much easier to calculate than its constituents ([8,9]), either by straightforward algebra or by using the theory of group characters. The sum may be replaced then by a simpler symbol:

$$(7) \qquad \sum_{\vartheta_1 \dots \vartheta_n} C_T(T_1 \dots T_n,\ \vartheta_1 \dots \vartheta_n)^2 = W_{\alpha\beta\gamma}(T) .$$

Namely: After summing over $\vartheta_1 \dots \vartheta_n$ there remains nothing else than the total number of isospin functions with eigenvalues T and ϑ, which one can construct by coupling n particles of isospin $T_1 \dots T_n$. This number only depends on T and on the numbers of particles with isospin $\frac{1}{2}$, 1, $\frac{3}{2}$, ... (It does not depend on ϑ!). Since we hardly will encounter particles with isospin $> \frac{3}{2}$, we denote by

$$(8) \qquad \left.\begin{array}{lll} \alpha & \text{the number of isospin } \frac{1}{2} & \text{- particles} \\ \beta & \text{» » » » } 1 & \text{»} \\ \gamma & \text{» » » » } \frac{3}{2} & \text{»} \end{array}\right\} \text{ occurring in the end-state.}$$

To stress again the formal relations between spin and isospin: In the angular momentum case even for larger particle numbers in the final state, their maximum added spin momentum is in general smaller than the total angular momentum. This implies, that all possible spin combinations, whatsoever their resultant may be, are allowed and practically equally probable. Assume for the moment that there are α, β, γ particles of spin (not isospin) $\frac{1}{2}$, 1, $\frac{3}{2}$ respectively and no others. Then the number of eigenfunctions of total spin S and third component (fixed but arbitrary) S_z would be just $W_{\alpha\beta\gamma}(S)$. But now all S and S_z are allowed, hence

$$\sum_{S, S_z} W_{\alpha\beta\gamma}(S) = \sum_{S} (2S+1) W_{\alpha\beta\gamma}(S)$$

is the number of states, and in fact the following relation holds:

$$(9) \qquad \sum_{S} (2S+1) W_{\alpha\beta\gamma}(S) = (2\cdot\tfrac{1}{2}+1)^{\alpha} \cdot (2\cdot 1+1)^{\beta} \cdot (2\cdot\tfrac{3}{2}+1)^{\gamma}$$

which is a special case of our spin-weight factor

$$\prod_j (2S_j+1)^{\sigma_j} .$$

([21]) F. Cerulus: to be published.

The meaning of F concerning the integration is: If F is just the volume elements $\mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_n$ (no integration), then we ask for the most detailed momentum-distribution. That is, of course, hopeless. The more modest one is, the more reliable the answer will be, since one knows very little about the matrix element squared.

If F is the whole momentum space, then we ask only for the probability to find a certain number of certain particles, which is the most modest question.

If F is the whole momentum space of $n-1$ particles, *i.e.* some $\mathrm{d}\boldsymbol{p}_j$ is taken outside the integral, that would yield the energy and angular distribution of the corresponding particle without regarding all the others.

Let us assume for a moment, we would try to achieve that latter description and let us re-order the particles such that the number of the particle in question is n:

$$(10)\quad P(F) = \mathrm{d}\boldsymbol{p}_n \cdot W_{\alpha\beta\gamma}(T) \cdot \left[\frac{V}{(2\pi)^3}\right]^{n-1} \cdot \frac{\prod (2S_j + 1)^{\sigma_j}}{\prod N_i!} \cdot$$

$$\cdot \int |\langle \boldsymbol{p}_1 \dots \boldsymbol{p}_n,\ m_1 \dots m_n | S | i \rangle_{T_1 \dots T_n}|^2\ \delta(E - E_f)\ \delta(\boldsymbol{P} - \boldsymbol{P}_f)\, \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_{n-1}\,,$$

is then the probability to find n particles with masses, spin and isospins as described, and any one of the particles of the same kind as particle n having a momentum in the cell $\mathrm{d}\boldsymbol{p}_n$ around $\boldsymbol{p}_n$. (It might be any one of them, since we have the factor $1/N_i!$).

f) Reduction of the set of possible statements because of ignorance concerning the matrix element. Angular distribution. – Now comes the most important step. We do not know the matrix element squared, but we may use the mean value theorem of integral calculus to take its mean value outside the integral. Omitting for a moment the other factors in front of the integral, we have then

$$(11)\qquad P(F) \sim \mathrm{d}\boldsymbol{p}_n \cdot S(\dots\dots) \cdot \int \delta(E - E_f)\ \delta(\boldsymbol{P} - \boldsymbol{P}_f)\, \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_{n-1}\,,$$

where the function S in front of the integral is the mean value. It depends no longer on $\boldsymbol{p}_1 \dots \boldsymbol{p}_{n-1}$, and therefore is a function

$$(12)\qquad S(E, T, \boldsymbol{P} | m_1 \dots m_n, T_1 \dots T_n, \boldsymbol{p}_n)\,.$$

The knowledge of this function would enable us to calculate the energy — angular distribution. There is no hope to do that in the near future.

We shall therefore restrict ourselves one step further by integrating over the angles of $\boldsymbol{p}_n$. This amounts to replace $\mathrm{d}\boldsymbol{p}_n$ by $4\pi p_n^2 \mathrm{d}p_n$ and to rewrite S using again the mean value theorem as

$$S(E, T, P \mid m_1 \dots m_n,\ T_1 \dots T_n, p_n)\,.$$

Further we introduce as an abbreviation

$$(13) \qquad \int \delta(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2})\, \delta(\boldsymbol{P} - \sum_{i=1}^{n} \boldsymbol{p}_i)\, \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_n \quad \varrho^*(E, m_1 \dots m_n, \boldsymbol{P})\,,$$

since it is, apart from unessential factors, the phase space density.

We now have [from here on we stay in the centre of mass system (CMS), *i.e.* $\boldsymbol{P} = 0$]

$$(14) \quad P(F) = W_{\alpha\beta\gamma}(T) \left[\frac{V}{(2\pi)^3}\right]^{n-1} \cdot \frac{\prod (2S_j + 1)^{\sigma_j}}{\prod N_i!} \cdot S(ET \mid m_1 \dots m_n,\ T_1 \dots T_n,\ p_n) \cdot$$
$$\cdot 4\pi p_n^2 \mathrm{d}p_n \cdot \varrho^*(E - \sqrt{p_n^2 + m_n^2},\ -\boldsymbol{p}_n)\,.$$

$P(F)$ gives here the momentum spectrum of particle n (in fact of anyone of that sort), averaged over all angles in CMS in the specified final state.

It is this quantity which we believe to be able to calculate to a good approximation and which the tables and figures given below refer to.

It is often said that the statistical model fails to explain anisotropies in CMS. From the point of view given here, this is a misunderstanding: This model here does not at all claim to answer to the question of angular distribution, rather it gives mean values over all angles.

Isotropy can, in the present model, not be concluded and there cannot be disagreement with experiments on this point, since we do not assume that (12) may be independent of the angle between $\boldsymbol{P}$ and $\boldsymbol{p}_n$. The use of the mean value theorem expresses our ignorance concerning the structure of (12). One may, however, take into account experimental facts about increasing isotropy with increasing multiplicity and say that a not too bad guess of the angular distribution in the lab. system may be found by *assuming* isotropy in CMS for certain secondary particles in certain sets of reactions (high multiplicity), and by transforming the spectra to the lab. system. But the justification cannot come from formula (14), it can come only from a knowledge of (12) or from experimental information.

How can we now calculate $P(F)$ according to (14)? There are two quantities which present difficulties, namely the functions S and ϱ^*. The latter one is mathematically well defined by (13) and can in principle be calculated. The integral can be evaluated in closed form only up to $n = 3$ and must be computed by numerical methods for $n > 3$. Various approximations have

been used in the literature. So far the only one which goes up to high particle numbers ($\lesssim 15$) and can be calculated within prescribed error limits (a few per cent, in reasonable time), is a Monte Carlo method developed by F. CERULUS and the author ([12]). It has been used in the present calculations and various other ones ([6,7,14,21]) by means of CERN's Ferranti-Mercury computer ([22]). Thus, we may consider the calculation of ϱ^* as settled.

2·2. – *Heuristic arguments leading to computable formulas: The model.*

a) Neglection of certain dependences of the matrix element. – It must be emphasized here that (14) is an exact formula as it stands. It is only now that we begin to use approximations; the correctness of which we do not know exactly and which can be justified only by the success. These approximations are drastic:

We assume

i) The dependences of S on T and $T_1 \dots T_n$ is negligible as compared with the dependences of $W_{\alpha\beta\gamma}(T)$ on these same variables.

ii) The dependence of S on the mass values is negligible as compared to the dependence of ϱ^* on them.

iii) The dependence of S on p_n is negligible against the variation of $p_n^2 \cdot \varrho^*$ with p_n.

If that is so, then $S(E, T | m_1 \dots m_n, p_n) \approx S(E, n)$. Integrating (14) over p_n does not affect $S(E, n)$ and the result would be

$$(15) \quad P(F') = S(E, n) \cdot W_{\alpha\beta\gamma}(T) \left[\frac{V}{(2\pi)^3}\right]^{n-1} \frac{\prod (2S_j + 1)^{\sigma_j}}{\prod N_i!} \cdot \varrho^*(E, m_1 \dots m_n, 0) .$$

Here $S(E, n)$ is (approximately) the mean value of the matrix element squared and all the other factors make up the volume of the total integration region (including sums). (15) is just the probability for finding n particles with masses, spins and isospins as specified.

b) Further information using unitarity and the inverse process. – For finding an information about $S(E, n)$ assume now, we intended to calculate the inverse reaction. The n particles specified are at $t = -\infty$ in a box of volume V as plane waves and at $t = +\infty$ just two protons fly off with high kinetic energy. Because of the unitarity of the S-matrix, the probability for this is given by the same matrix element squared as before, the difference being that initial and final states have interchanged their roles.

([22]) R. HAGEDORN: *A programme for calculating multiple phase space integrals by a Monte-Carlo method on the Ferranti Mercury Computer*, internal CERN report, 1959. Available on request.

Consequently integrating and averaging have also interchanged. Assume the two protons flying off in the same pure state as they come in the other experiment, then our present final state is pure: No summation. The initial state is a complicated mixture over which we must average. Obviously, $S(E, n)$ is that average [see (15) and the following text]. Thus for the inverse reaction

$$P_{\text{inv}}(F') = S(E, n) .$$

From purely geometrical reason, this probability must contain a factor $(\Omega/V)^{n-1}$ where Ω is a very small volume of the size of a nucleon and V is that of the normalization box. In fact, $\Omega \approx \Omega_0 \cdot (2M/E)$. Here $\Omega_0 = (4\pi/3)\lambda_\pi^3$; the factor $2M/E$ gives the necessary Lorentz-contraction of the off-flying two protons: Before they can fly off as required, all particles have to join in such a small flat volume. The exponent is $n-1$ only, since this may take place everywhere within V, in particular at the place of any one particle. Then only $n-1$ others must join there.

We now introduce approximation

iv) The dependence of $S(E, n)$ from E and n is mainly given by $(\Omega/V)^{n-1}$. This amounts to replacing V by Ω and omitting S in Eq. (14) and (15).

With this step, we reached a stage in which everything is ready for calculation. On the other hand, we have violated the rules from which we started and by which we agreed on defining initial and final states by means of plane waves. We took just now a classical picture and gave the particles' properties corresponding to a gas in thermal equilibrium enclosed in a box. Putting $S(E, n) \approx (\Omega/V)^{n-1}$ means in fact, that what we consider from now is a central collision. If namely, we would look for a peripheral collision and correspondingly here for n particles in a box joining into two nucleons, flying off with a large impact parameter, then certainly $(\Omega/V)^{n-1}$ would be too simple to describe that. The question of peripheral collisions is not yet well understood, although urgent and not hopeless. It will not be treated here.

c) Final state interactions and different primary interactions. – There are, however, two other points where we can improve this over simplified picture:

i) The notion of unstable particles meets in quantum theory with considerable difficulties. According to our heuristic approach, we should, however, include the possibility that the interaction between certain particles still goes on after the others are already free particles. This is obvious for bound states. If a deuteron is formed, it has to be considered as a free particle among the other free particles. There may be other states as the π-$\mathcal{N}$ isobar ($T = \frac{3}{2}$, $S = \frac{3}{2}$, $m = 1.316$) with a lifetime long enough to consider it as still existing

after its interaction with the other particles ceased to act. Then it has also to be considered as a free particle in what is called here the final state. Of course, in the observable final states, it has already decayed. So in principle, any such decaying state with *sufficient lifetime* must be treated as a particle ([3]). We include here only the above mentioned π-$\mathcal{N}$ isobar and call it N^*. Higher π-$\mathcal{N}$ resonances are today well established. We do not include them, since the results of the 6.2 GeV calculations ([7]) showed that without them, one comes near to the experiments. If these resonances would play an important role, this should clearly show up already at that energy (*). There are even isobars possible, which cannot be found directly in π-$\mathcal{N}$ experiments, *e.g.* such ones with $T = \frac{5}{2}$ or π-π isobars. We shall neglect all such possibilities (cf. the discussion on π-π isobars in the 6.2 GeV paper ([7])). The mentioned reasons may serve as an excuse for even not having made an attempt to incluse such states — simply because the computation would become unfeasibly long.

ii) It may happen that accidentally two particles come out near together and stay so for some time, because they have a small relative momentum. The probability for this can be calculated from phase space — at least, if the angular correlation between the two particles considered is approximately isotropic. If these two particles happen to be a particle-antiparticle pair, then they have a great chance for annihilation. This final state annihilation was treated in detail in the 6.2 GeV paper ([7]) and showed there a considerable effect both on the total number of antinucleons and on their spectrum. At the present energy the effect is small, since the probability for small relative momenta is small too. The effect on the total antinucleon number was calculated along the lines explained in ([7]) and it was found here that approximately 25% of the created pairs will undergo final state annihilation. The numbers given in the graphs below are corrected for this effect, assuming 25% annihilation for antinucleons and antihyperons, and adding the resultant π-mesons to those directly produced. It is not very likely that these corrections lie outside the deviations of the results from the truth. But, it is interesting to note that they are so small here, whereas at 6.2 GeV they had a large effect. This final state annihilation has nothing to do with the other one which happens in nuclei when the target is not hydrogen as supposed here.

iii) It is trivial that the decay of unstable particles with a very short lifetime (isobars, π^0, Σ^0) should be included, since they cannot be observed directly. Therefore, the spectrum of their decay products is calculated from their original spectrum. One assumes them to decay in their rest system into two particles with well defined four momentum and transforms back to the

(*) Apart from phase space reasons which make the higher isobars less important, also their life time seems to be small.

CMS. The result is (22)

$$(16)\qquad \begin{cases} w_\mu(\varepsilon) = \dfrac{m^*}{2k}\displaystyle\int_{y_0}^{y_1} \mathrm{d}y\, \frac{w^*(y)}{(y^2+2ym^*)^{\frac{1}{2}}}\,, \\ y_{\substack{1\\0}} = \dfrac{m^*}{\mu}\left[\dfrac{\varepsilon_\mu E}{\mu} \pm \sqrt{\left(\dfrac{\varepsilon_\mu E}{\mu}\right)^2 - (E^2+k_\mu^2)}\right] - m^*\,; \qquad E=\varepsilon+\mu\,. \end{cases}$$

Here $w^*(y)$ is the kinetic energy spectrum of the unstable particle of mass m^*, which decays as $m^* \to m+\mu$, $w_\mu(\varepsilon)$ is the kinetic energy spectrum of the particle with mass μ, ε_μ and k_μ are the total energy and the momentum of the particle μ seen from the rest system of the decaying particle m^*.

For the calculation of $w_m(\varepsilon)$ one has only to replace μ by m wherever it is relevant.

For μ or m or both going to zero, $y_1 \to \infty$ and y_0 tends to a well defined limit. These cases are included in the computing programme.

Formula (16) may be used also to calculate the spectra of hyperon and K decays (only two body decays!), which are not explicitly given in the results below.

iv) Even the strong interactions are not the same for all particles. The interaction volume Ω must in a way depend on both strength and range of interaction. The above mentioned magnitude of Ω fits quite well in the case of 2.75 GeV, where only pions are produced (6). When other particles occur also, one may look at Ω as a parameter, whose order of magnitude is given, but which within certain limits can be fitted to experiments. Even better were it to introduce a separate Ω for each kind of particle. This would allow one to consider $S(E, n)$ as still dependent on the masses of all particles produced. We would then have to replace Ω^{n-1} by a product $\prod \Omega_i^{n_i}$, i labelling the different kinds of particles. For the time being, it seems better to begin with less parameters, since one might find computational difficulties and also lose some insight by starting too generally.

We choose here three different Ω's (*)

Ω_π for $\mathcal{N}$, $\overline{\mathcal{N}}$, $\mathcal{N}^*$, $\overline{\mathcal{N}}^*$, π,

Ω_Y for hyperons and antihyperons (Y, $\overline{\mathrm{Y}}$),

Ω_K for K and $\overline{\mathrm{K}}$.

We have taken $(2\pi)^3 \cdot \Omega_\pi = (4\pi/3)\lambda_\pi^3 \cdot (2M/E)$ fixed and varied only the values of Ω_Y and Ω_K.

(*) The factors $1/(2\pi)^3$ have been absorbed into the Ω's, which from now on are equal to what $\Omega/(2\pi)^3$ was before.

This corresponds to assuming that the creation of $\mathcal{N}\overline{\mathcal{N}}$ pairs is a thing which, apart from phase space, is as probable as meson production is. The hyperons might have a different interaction and the K-mesons still another one. Again, apart from phase space, one had then the probability for creation of a $\overline{\mathrm{Y}}$-Y pair proportional to Ω_{Y}^2, for a $\mathrm{K}\overline{\mathrm{K}}$ pair proportional to Ω_{K}^2 and for Y-K associated production proportional to $\Omega_{\mathrm{Y}} \cdot \Omega_{\mathrm{K}}$. By variation of Ω_{Y} and Ω_{K} one can then try different aspects.

One might prefer to assign the Ω's to a different grouping of particles, *e.g.* Ω_1 for nucleons, pions, hyperons, Ω_2 for antinucleons and antihyperons, Ω_3 for K-mesons. The philosphy behind this would be that the production of anti-particles takes place in the core. In the 6.2 GeV (7) we found, however, the correct order of the antinucleon number treating pair creation with the same Ω as pion production, which supports our present choice. One may wonder, however, if the Ξ-particle fits into our scheme. Further experimental information seems necessary.

2·3. *Performance of the computation.* – Inserting the product of Ω's in (14) and (15) and absorbing one factor $[V/(2\pi)^3]^{-1}$ into the undetermined constant which, however, will be omitted altogether, we find for the probability of a certain set F of reactions

$$(17)\quad P(F) = W_{\alpha_i \beta_j}(T)\, \frac{\prod (2S_j + 1)^{\sigma_j}}{\prod N_i!} \cdot \Omega_\pi^{n_1} \Omega_{\mathrm{Y}}^{n_2} \Omega_{\mathrm{K}}^{n_3} \cdot \left(\frac{1}{2\pi}\right)^{3n} \cdot \varrho^*(E,\, m_1 \dots m_n,\, \boldsymbol{P} = 0)\,.$$

Here F means for instance the set of reactions $\mathrm{p} + \mathrm{p} \to \mathcal{N}^* + \overline{\mathcal{N}} + \Sigma + \mathcal{N} + \mathrm{K} + 2\pi$ (charges not specified). By prescribing such a set F, all variables in (17) are well defined (Ω_π, Ω_{Y}, Ω_{K} being chosen). For the momentum spectrum of particle i) (in fact of all particles of that kind), one has in the same reaction

$$(18)\quad w_i(p)\,\mathrm{d}p = \frac{P(F)}{\varrho^*(E,\, m_1 \dots m_n,\, \boldsymbol{P} = 0)} \cdot \\ \cdot 4\pi p^2\,\mathrm{d}p \cdot \varrho^*\left(E - \sqrt{p^2 + m_i^2},\, m_1 \dots \not{m}_i \dots m_n,\, -\boldsymbol{p}\right),$$

where $\not{m}_i$ means, that the mass m_i has been omitted (*). Of course

$$\int w_i(p)\,\mathrm{d}p = P(F)\,.$$

These expressions have been calculated for about 600 sets of reactions, such that all those which have a non-negligible probability contribute to the

(*) *i.e.* this latter ϱ^* is a $(n-1)$-particle phase space density.

given results. The second formula is calculated as a by-product in the calculation of the first one and needs no further computational work. Finally, the spectra are added together separately for each kind of particle, so that all reactions contribute with their proper weight. Then the spectra of unstable particles are split up into two, according to formula (16) and the spectra of the decay products are added where they belong to.

Finally everything is divided by $\sum P(F)$ so that the norm of each spectrum equals the number of particles of that kind produced in the average over all reactions. This number is printed out and afterwards all spectra are normalized to unity. It was more convenient in the calculations to express all spectra in kinetic energies, rather than in momenta.

The calculations of ϱ^* by a Monte Carlo method and of (17) and (18) by another programme has been described in detail in the papers ([12,16,17,22]).

3. – Critical discussion of the model: future possibilities.

In Section 2·1 we have derived an exact formula starting from the assumed knowledge of the S-matrix. In Section 2·2 we used some heuristic arguments to obtain formulas which can be evaluated numerically. We have there on the one hand

> made assumptions, which cannot be proved, but only be justified by the success, and which, though valid for certain energies, may break down in other cases,

and we have on the other hand

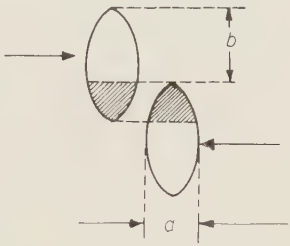
> by no means exhausted completely the knowledge, which from general principles, should be available on the scattering matrix.

This is a very unsatisfactory situation and one should look to improve it in both respects: by reducing the model features and by determining more of the structure of the function $S(E, T, m_1 \dots m_n, T_1 \dots T_n, p_n)$ in formula (14), using general knowledge of the S-matrix.

It seems, however, that the thing which one really *can* do presently, is just the opposite of what one should do. That is, it seems to be more hopeful to introduce even more details in the model under the guide of semi-classical pictures, then to determine the structure of S from the general properties of the S-matrix. Such classical pictures must be incorrect to a wide extent, but they can at least lead to an approach to the experimental facts.

3·1. *Non-central collisions and inelasticity.* – One of the weakest points is that one assumes in the present theory, that all the centre-of-mass energy is available for production of particles, whereas one sees easily that with increasing

impact parameter b finally a situation must be reached, where the incoming particles practically do not interact. This error comes in by replacing the function S by $[\Omega/V]^{n-1}$. On the other hand, one may guess that there is a range of impact parameters between 0 and $b_0 > 0$, such that all collisions with $b \leqslant b_0$ may be regarded as « central »: namely if the collision time is longer than the time necessary for spreading out the energy. A simple picture may lead to a crude estimate:



The radius of the nucleon is $\approx 1/\mu$ and the diameter of the Lorentz contracted nucleons in the direction of flight is

$$d \approx \frac{2}{\mu \cdot \gamma}; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{2} \,(M = 1!),$$

where E is the total CM-energy. The nucleons move almost with velocity of of light ($C = 1$). Consequently the time interval during which they are in direct contact in the shadowed region is $t \approx 2/\mu\gamma$. Assuming that the interaction through the meson cloud goes also with velocity of light, and that during the time of contact, this interaction must travel through the unshadowed region to and back at least once in order to spread the energy and destroy the identity of the nucleons, so that just a volume filled with energy develops — assuming that, we see that the condition for a « central collision » is

$$2b \leqslant t = \frac{2}{\mu\gamma},$$

or

$$b \leqslant b_0 = \frac{1}{\mu} \cdot \frac{2}{E}.$$

Another way of estimating how large b might be, is to say that during the time t the uncertainty relation allows a virtual particle (of mass $1/t$) to be born, which can travel at most a distance of length t before being re-absorbed. In order to spread the total energy over the whole region occupied by the

two nucleons, these virtual particles must reach at least the distance b; so b should be smaller than t.

The cross-section for « central collisions » will then be $\approx \pi b_0^2$ as compared to the geometrical cross-section $\approx \pi/\mu^2$. We have then

$$\text{1-st estimate} \qquad \frac{\sigma_{\text{central}}}{\sigma_{\text{geom.}}} \approx \frac{4}{E^2},$$

$$\text{2-nd estimate} \qquad \frac{\sigma_{\text{central}}}{\sigma_{\text{geom.}}} \approx \frac{16}{E^2}.$$

For this ratio, we find

$E_{\text{kin, lab}}$ [GeV]	2.75	6.2	25
1st estimate: $\sigma_{\text{centr}}/\sigma_{\text{geom}}$	0.4	0.23	0.07
2nd estimate: $\sigma_{\text{centr}}/\sigma_{\text{geom}}$	1	0.92	0.28

The present theory was said to be valid only for central collisions. Suppose for the non-central collisions, there would be another theory available, then one should multiply our results by $\sigma_{\text{centr}}/\sigma_{\text{geom}}$ and the other results by $1 - (\sigma_{\text{centr}}/\sigma_{\text{geom}})$ and superpose them. Strictly speaking, one should divide the geometric cross-section into ring-shaped regions and calculate everything separately for each region and finally add up. The model for this would be obviously to assume that the nucleons separate off before the equilibrium is reached and fly away in an excited state, the excitation energy and the remaining kinetic energy and the angle being in the statistical average a well defined, though unknown, function of the impact parameter b. After that the excited nucleons can be treated in their own centre of mass system again by the present theory. The results are transformed into the total centre of mass and integrated over b from b_0 to $1/\mu$, to which adds the « central » contribution. Such a model will imply, of course, further doubtful assumptions, but it contains on the other hand new parameters, which can (within certain obvious limits) be fitted to experiments (*e.g.* angular distribution). The amount of computation will be several times that of the present case, the latter being only a minor contribution, One might wonder now, how the results of the calculation at 6.2 GeV ([7]) could yield realistic particle numbers, as for instance, 2.9 pions in agreement with the experiment ([23]). There are two reasons:

i) The non-central collisions will also contribute to the particle production.

ii) The recently discussed π-π interaction with a resonant state, as well as higher isobars, which are neglected there, would open new channels for particle production.

([23]) R. M. Kalbach, J. J. Lord and C. H. Tsao, *Phys. Rev.*, **113**, 330 (1959).

It seems that the two things cancel: by neglecting the fact that only a small fraction of all collisions are central, one overestimates the particle production. By neglecting other possible (though not well known) channels, one underestimates the particle production. The agreement with the experiments at 2.75 GeV ([6]) and 6.2 GeV ([7]) seems to indicate this compensation ([23,24]). It is not safe to extrapolate that to our energy, but for the moment one has nothing better. That this extrapolation might be not so bad, follows from Figs. 18 and 16, where we plot the calculated yield of newly produced charged particles and the ratio of non-pionic charged particles to all charged particles versus energy. For 2.75 and 6.2 GeV these numbers agree well with the experiments ([23,24]) for 10 GeV the particle production seems to be somewhat overestimated ([25]) and for very high energies the curves seem to tend to the cosmic ray data.

As long as no theory exists which can deal with non-central collisions, one might take our present results together with a cross-section, which is $\frac{1}{4}$ to 1 times the geometric one.

Intimately connected with the question of non-central collisions is the inelasticity. One should introduce two different notions of inelasticity, the one coinciding with what experimentalists define as inelasticity and which can be measured as the ratio of the total CM-energy of all produced particles divided by the total kinetic CM-energy before the collision:

$$K = \frac{E\ (\text{produced particles})}{E - 2}\,.$$

It is obvious that the present theory allows to calculate this quantity and it yields always $K < 1$, since, though all collisions are treated as central, the nucleons carry away kinetic energy and there is even a non-vanishing probability for no new particle being produced. So for each of the about 600 final states computed here, one could also calculate the observable inelasticity K and there will be reactions, where K is zero or very small.

There is, however, no reason to believe that this calculated inelasticity would agree with the measured one, except for low energies: at 2.75 the calculated and measured inelasticity are close together. The above considerations on non-central collisions show, why for high energies we should not expect to find agreement: clearly only a part of the total kinetic CM-energy will be offered to particle production, the rest will remain as the kinetic energy of

([24]) M. M. Block, E. M. Harth, W. B. Fowler, R. P. Shutt, A. M. Thorndike, W. L. Whittemore, V. T. Cocconi, E. Hart, E. C. Fowler, J. D. Garrison and T. W. Morris: *CERN Symposium, Geneva* (1956).

([25]) V. Veksler: *High Energy Conference, Kiev* (1959).

the excited nucleons. Therefore one should introduce the concept of « offered inelasticity »:

$$k = \frac{E\ (\text{excitation})}{E - 2}\ .$$

This quantity is defined only with respect to a definite model. In our case (only « central collisions » considered) it is of course $= 1$. If one adopts for the non-central collisions the two centre model (which has been proposed and investigated for the case of « jets » by many authors), then k will be some function of the impact parameter b: in the most crude picture it would be proportional to the volume of the overlap region. If one then applies the present theory to the two centres, one can calculate for each of them analogue to K, namely the ratio

$$K_i' = \frac{E_i\ (\text{produced})}{E_i\ (\text{excitation})}\ ; \qquad i = 1, 2.$$

Transforming to the total CM and using

$$E_1\ (\text{excit.}) + E_2\ (\text{excit.}) = k \cdot (E - 2)$$

one can express the observable K by means of k and the K_1', *i.e.* by quantities which are uniquely defined by the adopted model. This would yield a further check of such models.

Apart from the existence of « jets » with strongly peaked CM angular distribution, there is another indication that such a two centre model might be successful: if one calculates with the present theory the mean kinetic energies of all particles, then one finds them to be nearly independent of the primary energy (see the calculations at 2.75 GeV ([6]), 6.2 ([7]) and the present results) and they lie always in the range between about $250 \div 450$ MeV (except for particles just above production threshold). The same order of magnitude is found in cosmic ray experiments for the average transversal kinetic energy up to very high primary energies. Since one would apply the present theory to each centre separately and then assume isotropic distribution in their centres-of-mass, it is obvious that the calculated transversal kinetic energy would agree with the experimental one.

3·2. *Strange particles.* – The adopted three-Ω model is certainly too crude to give a detailed picture of strange particle production. For the actual calculation, we choose two different versions: the Ω_π for pions and nucleonic particles is always constant and has its « natural » values $(4\pi/3)\lambda_\pi^3 \cdot (1/2\pi)^3$. The Ω_K is always varied between $\frac{1}{50}\,\Omega_\pi$ and Ω_π. The Ω_Y for the hyperons has

been kept constant and equal to Ω_π in the first version (I) and has been varied linearly together with Ω_K in version (II):

(I) $\Omega_Y = \Omega_\pi$; Ω_K variable,

(II) $\Omega_Y = \frac{4}{9}\Omega_\pi + \frac{5}{9}\Omega_K$.

It is not meant that these choices have any particular significance — we rather choose them because it was not possible to vary Ω_Y and Ω_K independently. However, the choice $\Omega_Y = \Omega_K$ as taken in the 6.2 GeV case [7] seemed not very promising. On the other hand, a $\Omega_Y < \Omega_\pi$ is not unreasonable. If $\Omega_K = \frac{1}{10}\Omega_\pi$ should be right, then (I) and (II) reduce to $\Omega_Y = \Omega_\pi$ and $\Omega_Y = \frac{1}{2}\Omega_\pi$. The spectra are not very sensitive to the versions and the value of Ω_K. Those given in the figures are for (I), $\Omega_K = \frac{1}{10}\Omega_\pi$. (They were compared to (II), $\Omega_K = (1/3.4)\Omega_\pi$ and equal to them within the accuracy of drawing).

It should be mentioned that both (I) and (II) allow reactions of the kind

$$\pi + \pi \to \pi + \Sigma + \Lambda + \Xi ,$$

without producing K-mesons, even for $\Omega_K = 0$. (This is also the reason, why for (I), Ω_K small, the production of $\bar{\Xi}$ is slightly higher than the Ξ-production). Unless one introduces a fourt Ω_Ξ for the Ξ, this is unavoidable. But, since in fact the K-interaction is not zero, one might take this not too seriously. The results on Ξ and $\bar{\Xi}$ production should, however, be considered as much less reliable as all the others.

3·3. *Spectra of the anti-particles.* – Though the particle numbers have been roughly corrected for final state pair annihilation, this has not been done for the spectra — even not for those of the anti-particles. The amount of further calculation (see the 6.2 GeV paper [7]) would have been enormous, since at 25 GeV in most reactions pair production occurs. But, since the kinetic energies of the antiparticles are large now (at 6.2 GeV one is just above threshold), the effect will hardly justify the effort.

3·4. *Phase space integrals.* – The method used here to calculate the phase space integrals and spectra is a quite complicated Monte-Carlo method [12], which needs an electronic computer [22]. The integrals can, however, also be calculated analytically for two limiting cases, namely for unrelativistic and for zero-mass particles. It has been frequent use to consider nucleons and heavier particles as unrelativistic and pions as ultra-relativistic (massless) and then either to assume that for each of these two groups the total momentum vanishes or—for a better approximation—to fold the two phase space integrals

in a further integral:

$$\varrho^*(E,\ m_1 \dots m_n,\ \boldsymbol{P}) \simeq \int \varrho^*_{\text{heavy masses}}(E-\varepsilon,\ \boldsymbol{P}-\boldsymbol{p}) \cdot \varrho^*_{\text{zero masses}}(\varepsilon, \boldsymbol{p})\, \mathrm{d}\varepsilon\, \mathrm{d}\boldsymbol{p}\ .$$

It has been shown ([12]) that this can lead to serious errors. The reason is clear: the average kinetic energy turns out to be about 300 MeV per particle nearly independently of the primary energy. Therefore neither the heavy particles are non-relativistic, nor the mesons ultra-relativistic. For K-particles, it is even worse. Since according to Figs. 1-8, the width of the spectra is of the order of the mean kinetic energy, it is clear that important contributions to ϱ^* come from energy-regions where the above mentioned approximation, which is already bad for the mean energies, becomes intolerable. There is, of course, some mutual compensation, but in high dimensional space this is hard to estimate. The constancy of the mean kinetic energy shows that this situation will be the same for all primary energies.

The presently used Monte-Carlo method is free from such errors and allows to treat all masses correctly. The unavoidable statistical errors (which for each *single* reaction can be prescribed and have been chosen to be $< 5\%$) average out over the more than 600 calculated reactions. Even in the spectra there remained hardly any fluctuations. In the present case, the adopted method for the calculation of phase space integrals can therefore be considered as exact (in the 6.2 GeV case ([7]) only about 50 reactions contributed and some fluctuations remained).

* * *

I am deeply indebted to many people who helped me in the preparation of this paper: Dr. H. Faissner of CERN, prepared and watched at the computer, a considerable part of the Monte-Carlo calculations and took part in many discussions; Drs. F. Cerulus, B. d'Espagnat, J. Prentki and Y. Yamaguchi, all of CERN, helped me with many interesting talks.

The whole work would have been impossible without the constant collaboration of the engineers and the tape punching group of the CERN-Computer.

Appendix

Results.

The results are given in three groups of figures:

i) Figures 1-8 show the kinetic energy spectra of the different kinds of particles as

$$\frac{\text{mean particle number per 100 MeV energy interval}}{\text{mean particle number}} = \frac{1}{\langle n \rangle} \cdot \frac{\langle \Delta n}{\langle \Delta \varepsilon} \ .$$

The spectra are normalized to unity. In order to obtain the number of particles per energy interval, one has therefore to multiply by the mean particle number $\langle n \rangle$, which is found in the second group of graphs, namely

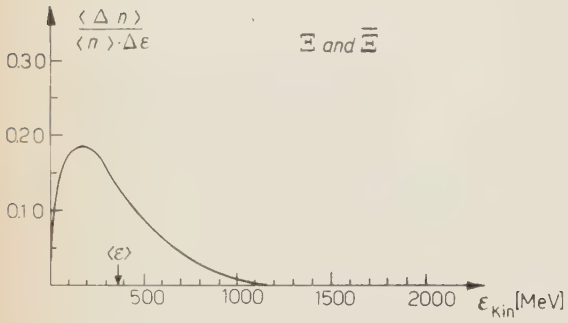


Fig. 1. – Spectrum of Ξ and $\overline{\Xi}$.

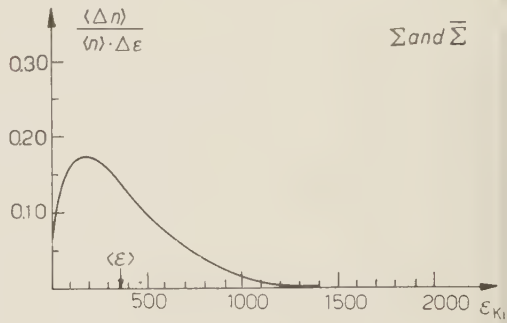


Fig. 2. – Spectrum of Σ and $\overline{\Sigma}$.

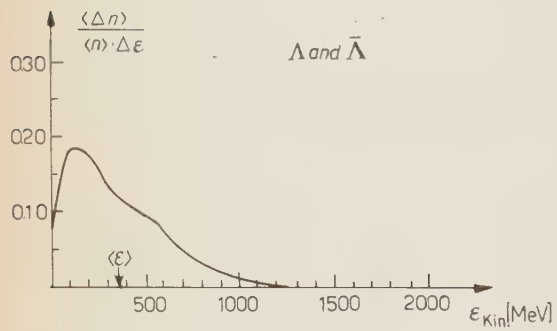


Fig. 3. – Spectrum of Λ and $\overline{\Lambda}$.

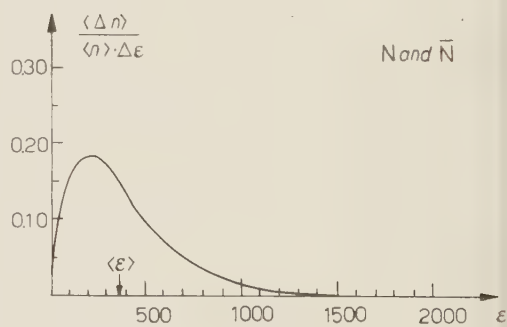


Fig. 4. – Spectrum fo $\overline{\mathcal{N}}$ and $\mathcal{N}$.

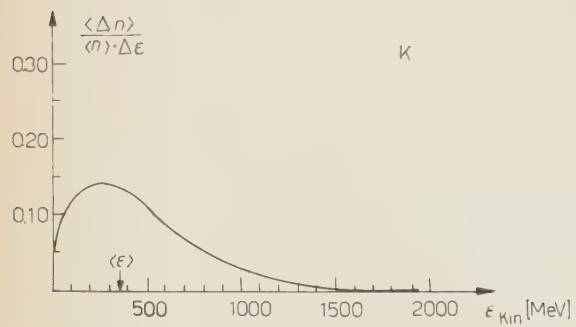


Fig. 5. – Spectrum of K.

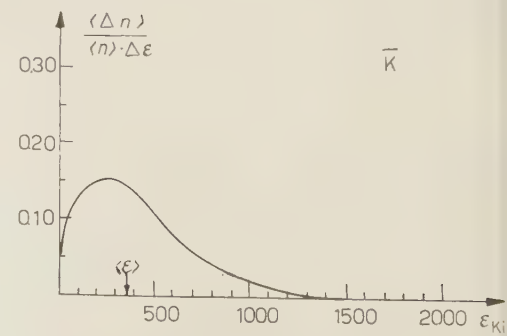
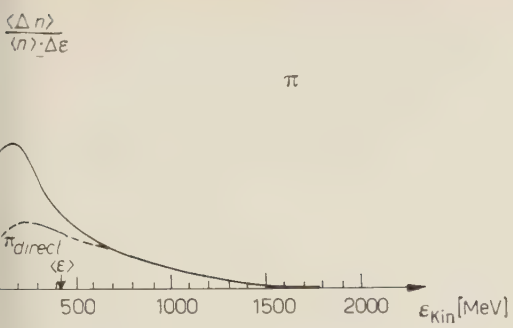


Fig. 6. – Spectrum of $\overline{\mathrm{K}}$.



. 7. – Spectrum of π. The dotted line ws what the spectrum would be without ing into account the $(\frac{3}{2};\frac{3}{2})$ π-$\mathcal{N}$ isobar.

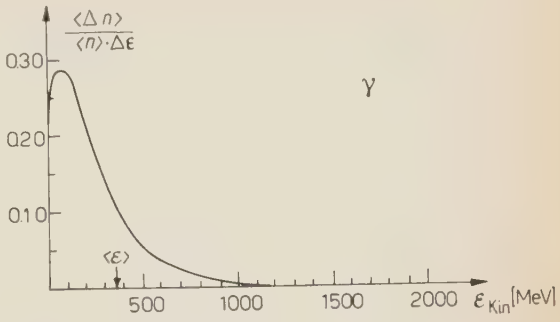


Fig. 8. – Spectrum of γ only from $\Sigma^0 \to \Lambda^0 + \gamma$ and $\pi^0 \to 2\gamma$.

ii) Figures 9-14, which show the mean particle numbers $\langle n \rangle$ per collision (total, *i.e.* summing over all charges, including neutral. They are corrected for final state annihilation.) as function of Ω_K in the two versions

(I) $\qquad \Omega_Y = \Omega_\pi = \text{const}\,; \quad \Omega_K$ variable,

(II) $\qquad \Omega_\pi = \text{const}\,; \quad \Omega_Y = \frac{4}{9}\Omega_\pi + \frac{5}{9}\Omega_K\,; \quad \Omega_K$ variable.

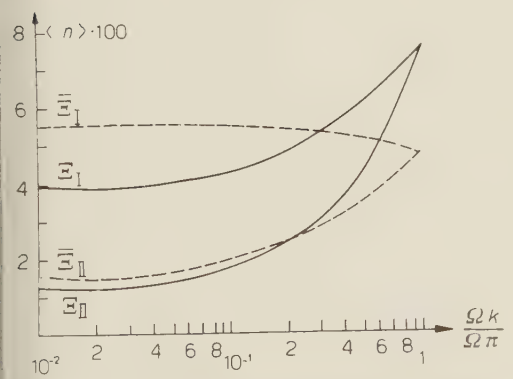


Fig. 9. – Ξ and $\bar{\Xi}$-multiplicity (rather doubtful).

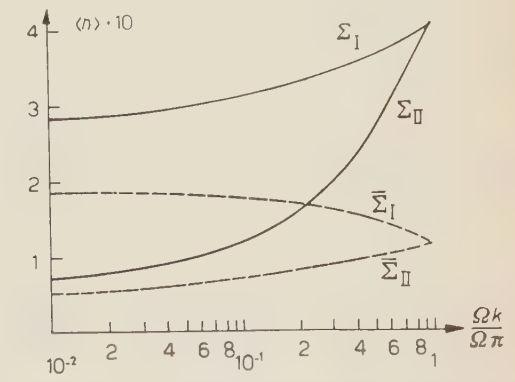


Fig. 10. – Σ and $\bar{\Sigma}$-multiplicity (including Σ^0 and $\bar{\Sigma}^0$).

One may expect reasonable predictions (probably except Ξ and $\bar{\Xi}$ particles) for

$$\Omega_Y \approx (\tfrac{1}{5} \text{ to } \tfrac{1}{15})\cdot\Omega_\pi \text{ in (I)}$$

and

$$\Omega_K \approx (\tfrac{1}{2} \text{ to } \tfrac{1}{6})\cdot\Omega_\pi \text{ in (II)}.$$

The spectra refer to these regions (they are rather insensitive against change of Ω_K).

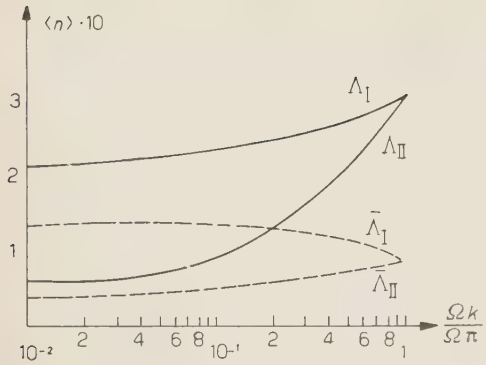


Fig. 11. – Λ and $\bar{\Lambda}$-multiplicity.

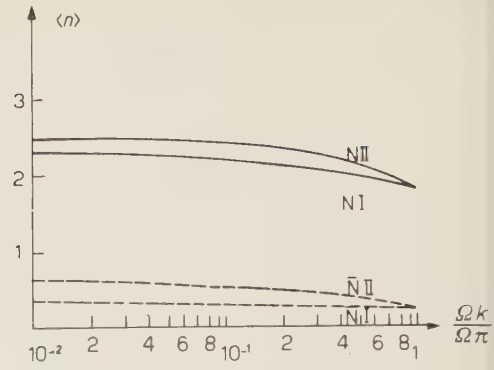


Fig. 12. – $\mathcal{N}$ and $\overline{\mathcal{N}}$-multiplicity.

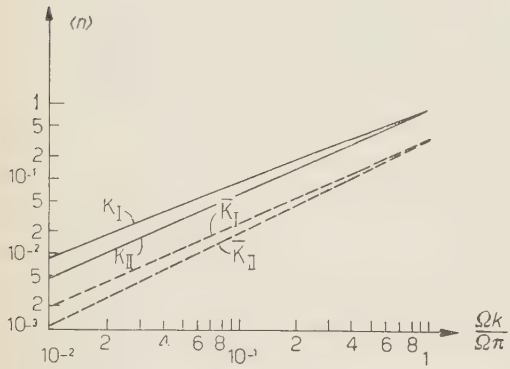


Fig. 13. – K and $\bar{\text{K}}$-multiplicity (the only strongly varying multiplicity).

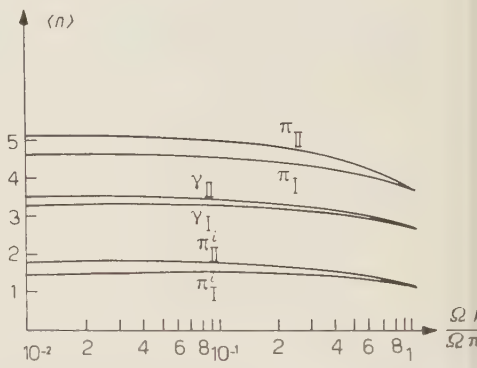


Fig. 14. π, π^i and γ-multiplicities; π^i is th pion which comes from the $(\frac{3}{2};\frac{3}{2})$ π-$\mathcal{N}$ isobar

iii) Figures 15-18 show multiplicities and ratios of pions to other particles as function of Ω_K and as functions of energy.

As to the spectra, one finds that the particles and antiparticles have the same spectra. This comes from the fact that the spectra are mainly given by the phase space part and in these particles and anti-particles are not distinguished. Though they enter in the total spectrum with somewhat different weights, this does not exceed the accuracy of drawing, since in most of the 25 GeV reactions, anti-particles are produced. Only the K-mesons show a small difference.

As was emphasized repeatedly in this paper, the spectra are meant as averaged over all CM-angles. For pions, K-mesons and anti-baryons, one may, however, *assume* approximate isotropy and estimate the lab-angular distribution, simply by transforming them. The same holds for the γ-spectrum, which comes exclusively from the decays $\Sigma^0 \to \Lambda^0 + \gamma$ and $\pi^0 \to 2\gamma$. The nu-

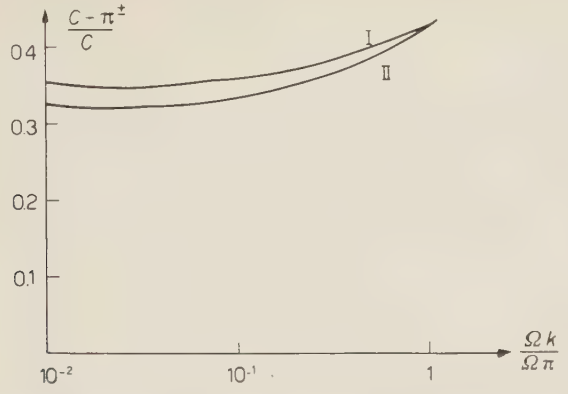


Fig. 15. – C is the total number of charged particle. $\pi^{\pm}$ the number of charged pions. The plot shows the ratio of non-pionic charged particles to all charged particles as a function of Ω_K.

cleons will probably be so strongly peaked in the forward and backward direction, that transforming to the lab system is not very meaningful.

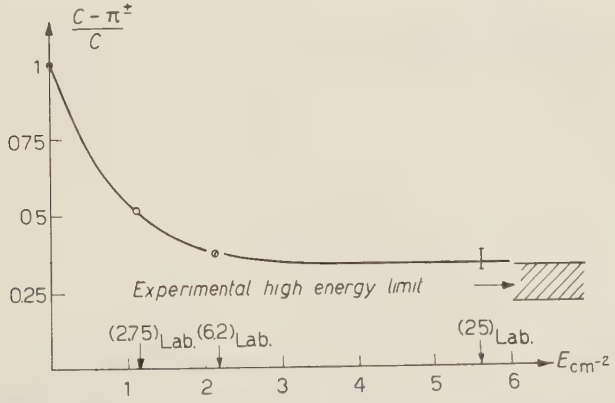


Fig. 16. – The same quantity as in Fig. 15 is shown here, as a function of the total kinetic CM-energy before the collision. At 2.75 GeV (lab) ([6]) and 6.2 GeV (lab) ([7]). the values can be shown to agree with the experimental values ([23,24]) (by somewhat lenghty arguments, since they are not directly published as experimental values). At our present energy, the bar indicates the values compatible with reasonable Ω_K (see Fig. 15). The curve is drawn tentatively through the given points. It tends to the experimental high energy limit ([26-28]).

([26]) F. A. Brisbout, C. Dahanayake, A. Engler, Y. Fujmoto and D. H. Perkins: *Phil. Mag.*, **1**, 605 (1956).

([27]) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds *Phil. Mag.*, **3**, 237 (1958).

([28]) E. Lohrmann and M. W. Teucher: *Phys. Rev.*, **112**, 587 (1958).

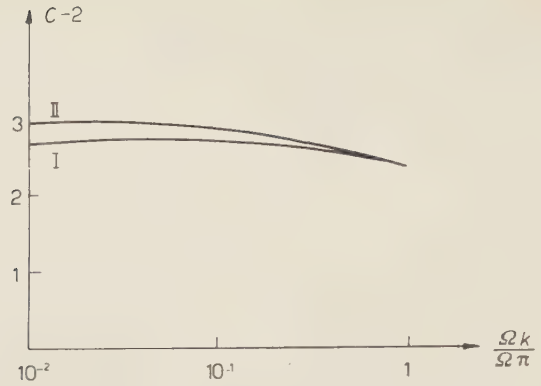


Fig. 17. – The number of charged particles minus two as function of Ω_K.

As to the multiplicities, Figs. 9-14, one sees that except those for the K-mesons, they do not very much vary (only the K-multiplicities are in loga-

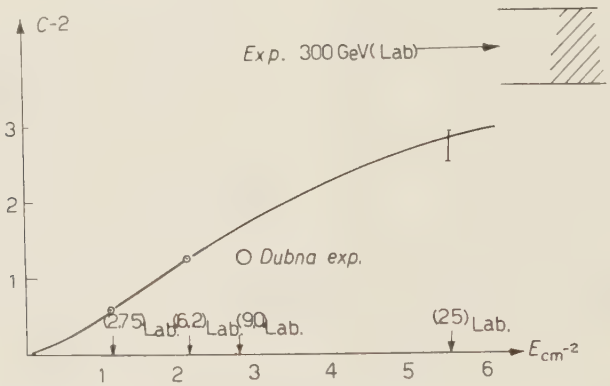


Fig. 18. – C-2 as a function of the total kinetic CM-energy before the collision. At 2.75 GeV lab) ([6]) and 6.2 GeV (lab) ([7]), this agreees again with the experiments ([23,24]) (by indirect arguments as in Fig. 16). The bar at 25 GeV (lab) indicates the values compatible with reasonable Ω_K (see Fig. 17). The bubble at 9.0 GeV (lab) represents the measurements at Dubna ([25]). For high energies, the curve tends to estimated cosmic ray data (M. Teucher, Chicago; private communication).

rithmic scale). The number of particles of a given charge are found approximately by dividing the n's by $2T+1$, where T is the total isospin of the particle in question.

ERRATUM:

In the article the final state annihilation has been taken into account in a very rough way as described there. Strangeness conservation had not been imposed on that process, but it has turned out that its effect cannot be neglected: in each case where a hyperon or antihyperon annihilates, a K or a K-meson respectively must remain. This will increase the number of both of them, but more the K than the $\bar{\mathrm{K}}$. This taken into account, the mean particle numbers fulfil strangeness conservation, which in fact is not true for those given in the article.

One may correct this in the article by a parallel shifting of the K and $\bar{\mathrm{K}}$ curves of Fig. 13.

The pair of full lines ($\mathrm{K_I}$ and $\mathrm{K_{II}}$) is shifted such that $\mathrm{K_I}$ passes through the point

$$\frac{\Omega_{\mathrm{K}}}{\Omega_\pi} = 0.2; \qquad \mathrm{K_I} = 3.2 \cdot 10^{-1},$$

and the pair of dashed lines ($\bar{\mathrm{K}}_{\mathrm{I}}$ and $\bar{\mathrm{K}}_{\mathrm{II}}$) is shifted such that $\bar{\mathrm{K}}_{\mathrm{I}}$ passes through the point

$$\frac{\Omega_{\mathrm{K}}}{\Omega_\pi} = 0.2; \qquad \bar{\mathrm{K}}_{\mathrm{I}} = 8.0 \cdot 10^{-2},$$

which means that the predicted number

of K is roughly two times larger and of $\bar{\mathrm{K}}$ is roughly 1.6 times larger,

than given in the article.

RIASSUNTO (*)

Si dà una nuova derivazione del modello statistico per la produzione di particelle, a partire dalla teoria della matrice S. Quantunque le formule finali siano essenzialmente quelle già usate in altre pubblicazioni recenti, questa derivazione mostra chiaramente dove si introducano ragionamenti euristici e quali siano i punti deboli. Si vede anche che, contrariamente all'opinione diffusa, l'isotropia angolare rispetto al sistema del c.m. non è nè una conseguenza, nè un presupposto della teoria statistica; i risultati di tale teoria si riferiscono semplicemente a medie rispetto a tutti gli angoli nel sistema del c.m. Una discussione critica dimostra, tuttavia, che anche la forma presente della teoria si applica solo ad urti centrali (in senso lato) e deve essere estesa agli urti non centrali, se si vogliono dati molto dettagliati. Si espongono argomentazioni che dimostrano come ciononostante la teoria degli urti centrali può dare risultati in accordo con gli esperimenti per molte grandezze interessanti (come dimostra l'esperienza nel caso di 2.75 GeV e di 6.2 GeV). Si propone una più precisa differenziazione del concetto di inelasticità. Si danno, sotto forma di curve, gli spettri e i numeri di produzione media per urti p-p a 25 GeV.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XV, N. 3 1° Febbraio 1960

Note on the High-Energy Tail of the Pion and γ-Spectra in p-p Collisions at 25 GeV.

R. Hagedorn

CERN - Geneva

(ricevuto il 6 Novembre 1959)

Summary. — The energy distribution of high energy pions emerging from p-p collisions at 25 GeV is estimated using the statistical model. Indications for the calculation of the γ-spectrum from these data are given.

For some experiments high energy pions and γ-rays may be useful. In a paper on statistical theory applied to 25 GeV p-p collisions [1] energy spectra and particle numbers were given as curves. The limited accuracy of drawing caused the high energy tail of the spectra to be suppressed. Of course in the numerical results on which the paper [1] is based, much more information is contained and can be taken out if necessary. The high energy tail of the pion spectrum seems to be such an information and the present note gives the details on it.

It must be mentioned that the Table given below represents only an estimate. The contributions to the tail come from a few reactions only, namely those in which not many particles are produced. The accuracy of the phase-space integrals in each single reaction was prescribed to be $\leqslant 5\%$. A spectrum (whose sum equals the phase space integral for the reaction in question) contains 31 numbers, hence its statistical accuracy is $\sim 25\%$ for each number. The data given in the Table below are found by graphical smoothing out of

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[1] R. Hagedorn: *Nuovo Cimento*, **15**, 434 (1960).

histograms in which each number was given within $\pm 10\%$. Therefore the numbers in the Table are correct within 5% to 10% as far as the underlying theory is right.

The Table gives the *absolute* number of pions per kinetic energy interval of 100 MeV. In the paper [1] all spectra were normalized to unity, therefore the low energy part of the Table differs from Fig. 7 of the paper [1] by the mean pion number, which was chosen here $\langle n \rangle_\pi = 4.5$.

TABLE I. – *Absolute number of pions per* 100 MeV.

E_{kin} GeV	Total π	π^+	π^0	π^-
1.1	$7.8 \cdot 10^{-2}$			
1.2	$5.8 \cdot 10^{-2}$			
1.3	$4.2 \cdot 10^{-2}$			
1.4	$3.0 \cdot 10^{-2}$			
1.5	$2.1 \cdot 10^{-2}$	roughly $\frac{1}{3}$ of total π	roughly $\frac{1}{3}$ of total π	roughly $\frac{1}{3}$ of total π
1.6	$1.6 \cdot 10^{-2}$			
1.7	$1.1 \cdot 10^{-2}$			
1.8	$7.5 \cdot 10^{-3}$			
1.9	$5.3 \cdot 10^{-3}$			
2.0	$3.5 \cdot 10^{-3}$			
2.1	$2.3 \cdot 10^{-3}$			
2.2	$1.5 \cdot 10^{-3}$			
2.3	$8.7 \cdot 10^{-4}$			
2.4	$4.9 \cdot 10^{-4}$			
2.5	$3.0 \cdot 10^{-4}$			
2.6	$1.6 \cdot 10^{-4}$			
2.7	$9.0 \cdot 10^{-5}$	$4.3 \cdot 10^{-5}$	$2.7 \cdot 10^{-5}$	$2.0 \cdot 10^{-5}$
2.8	$4.8 \cdot 10^{-5}$	$2.3 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$	$1.0 \cdot 10^{-5}$
2.9	$2.5 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$	$8.0 \cdot 10^{-6}$	$5.0 \cdot 10^{-6}$
3.0	$1.2 \cdot 10^{-5}$	$5.3 \cdot 10^{-6}$	$4.2 \cdot 10^{-6}$	$2.5 \cdot 10^{-6}$
3.1	$2.4 \cdot 10^{-6}$	$1.5 \cdot 10^{-6}$	$0.9 \cdot 10^{-6}$	0
3.2	0	0	0	0

The charge analysis [2] has been carried through only for the extreme tail, since there one-, two-, and three-pion production contributes. The rule that each charge state obtains approximately $\frac{1}{3}$ of the total number becomes very inaccurate there as the Table shows. For the lower energy regions this approximation becomes tolerable and the long calculations for charge analysis are not justified. It may be interesting to note, that even to the high energy tail, as given here the π-$\mathcal{N}$-isobar $(\frac{3}{2}; \frac{3}{2})$ gives important contributions.

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[2] See for instance: Multiple meson production from p-p collisions at 2.75 GeV according to a statistical theory: by F. CERULUS and R. HAGEDORN, CERN 59-3.

The γ-spectrum from π^0 decay has not been calculated explicitly. This can be done by using Equ. (16) of the paper [1] and taking the limits of the integral for $m = \mu = 0$. For most purposes however one will need only pions going in the forward direction and then emitting a γ in the forward direction again.

Of course, the data apply to the CM system and must still be transformed to the lab. system. Though it is not reliable, one may assume isotropy of the pions in CM and get at least correct orders of magnitude.

* * *

I thank Dr. F. Cerulus (CERN) for giving me some coefficients for the charge analysis.

RIASSUNTO (*)

Si stima la distribuzione energetica dei pioni di alta energia emergenti da collisioni p-p a 25 GeV servendosi del modello statistico. Si danno indicazioni per il calcolo dello spettro γ a partire da questi dati.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XV, N. 3 1° Febbraio 1960

Pion Cloud Effects in Pion Production Experiments.

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(ricevuto il 24 Novembre 1959)

Summary. — A simple model is proposed for the study of single pion production in collisions of a generic particle « a » on a target nucleon. The model consists essentially in neglecting all interactions but the one between the incoming particle and a single pion of the nucleon cloud. Application to a recent pion production experiment in proton-proton collision by the Birmingham group (1) gives support to the plausibility of the idea. Another application to the experiment by the Bologna group (2) gives qualitative evidence of pion-pion interaction. Experimental data suggest a pion-pion cross section of the order of ten millibarns. Finally an experiment is proposed which could give more detailed information on pion-pion interaction.

1. – Introduction.

The possibility that the meson cloud plays an important role in experiments of scattering on nucleons has already been recognized. It has also been remarked that from pion production experiments it should be possible to obtain some information regarding the interaction of the incoming particle with the virtual pions and methods have been proposed to obtain the cross-section of this interaction (3,4). Yet practical difficulties, depending also from the great

(*) Now at CERN, Geneva.

(1) A. P. Batson, B. B. Culwick, J. G. Hill and L. Riddiford: *Proc. Roy. Soc.*, **251**, 218 (1959).

(2) V. Alles-Borelli, S. Bergia, E. Perez-Ferreira and P. Waloschek: *Nuovo Cimento*, **14**, 211 (1959).

(3) C. Goebel: *Phys. Rev. Lett.*, **1**, 337 (1958).

(4) G. F. Chew and F. E. Low: *Phys. Rev.*, **113**, 1640 (1959).

statistic needed, have, until now, made impossible the application of these methods.

In what follows we want to examine in a semi-phenomenological fashion two recent experiments on pion production.

The first one, by A. P. BATSON, B. B. CULWICK, J. G. HILL and L. RIDDIFORD [1], regards pion production in proton-proton collisions. We will be particularly concerned with events of the kind

$$p + p \to p + n + \pi^+ .$$

The second one, by V. ALLES-BORELLI, S. BERGIA, E. PEREZ-FERREIRA and P. WALOSCHEK [2], regards pion production in negative pion-proton collisions.

In this analysis we will use a model consisting essentially in neglecting both direct interaction and more of one pion exchanges between the incoming particle and the nucleon. This model will be presented in Section 2, together with the experimental and theoretical justifications that can be given. In Sections 3 and 4 it will be applied to the quoted experiments of pion production in proton-proton and pion-proton collisions, respectively, with the aim of obtaining some information on proton-pion and pion-pion interactions. We will see that the obtained pion-proton cross-section agrees reasonably well with the experimental one, and this fact will give further support to the method.

In the case of the pion-pion interaction a qualitative evidence will be found, but only the order of magnitude of the cross-section given, the desired effect being partially obscured by the strong final state 33 interaction. However an experiment will be proposed in which this interaction should be almost negligible which could in principle give not only the correct mean value of the pion-pion cross-section, but also the position of an eventual resonance.

2. – The model.

The experiments on high energy proton-proton and pion-proton elastic scattering show a pronounced diffraction-like forward peak, which gives for the nucleon radius a value in every case very similar to the Compton wave length of the pion. This is true, in particular, for the two experiments we want to examine in the next sections ((see [1]) and [5]). This fact strongly suggests that the interaction responsible for the absorption of the incoming particle from the elastic channel takes place also with the pion cloud of the

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[5] V. ALLES-BORELLI, S. BERGIA, E. PEREZ-FERREIRA, A. MINGUZZI-RANZI and P. WALOSCHEK, to be published.

target nucleon; this conclusion, obvious in the case of nucleon-nucleon scattering, suggests the existence of pion-pion interaction in the case of pion-nucleon scattering.

On the other hand the mean number of pions in the nucleon cloud, calculated both with the static theory ([6]) and with a relativistic theory of intermediate coupling ([7]), turns out to be about one. Furthermore KOBA and TAKEDA ([8]), taking this number equal to 1.3, have been able to explain the multiplicity of pions produced in proton-antiproton annihilations. All this suggests the interaction of the incoming particle with more than one virtual pion to be an unlikely event.

In the following we will make the explicit assumption that the only important contribution to the interaction between the incoming particle and the target nucleon responsible for single pion production comes from diagrams with only one virtual intermediate pion. In Figs. 1-*a*, *b* are drawn such dia-

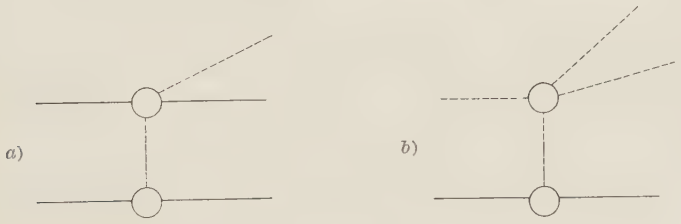

Fig. 1.

grams in the cases of nucleon-nucleon and pion-nucleon scattering. Full and dashed lines represent nucleons and pions respectively. The matrix elements corresponding to such graphs are, however, not exactly calculable with present techniques, if one wants to take into account all possible vertex corrections and all self-energy processes of the intermediate pion. A form valid only when the square of the four-momentum of the spectator nucleon, approaches the unphysical limit $-\mu^2$, μ beeing the mass of the pion, has been calculated by CHEW and LOW ([1]) and it is, after integration over all variables but Δ^2 and w^2, w beeing the total energy of all the final state particles except the recoil nucleon in their c.m. system,

$$\frac{\partial^2\sigma}{\partial\Delta^2\,\partial w^2} \to \frac{f^2}{2\pi}\frac{1}{q_{aL}^2}\frac{\Delta^2/\mu^2}{(\Delta^2+\mu^2)^2}\,\sigma_{\pi a}(w)m_a q'_\pi(w)\,, \tag{1}$$

([6]) S. FUBINI: *Nuovo Cimento*, **3**, 1425 (1956).
([7]) G. TAKEDA: *Phys. Rev.*, **95**, 1078 (1954).
([8]) Z. KOBA and G. TAKEDA: *Progr. Theor. Phys.*, **19**, 269 (1958).

where f^2 is the renormalized coupling constant for the emission of the virtual pion by the nucleon in the « lower » vertex—we make reference to Fig. 1— q_{aL} and m_a are laboratory momentum and mass of the incoming « a » particle, $\sigma_{\pi a}(w)$ and $q'_{\pi}(w)$ are the total cross-section and laboratory momentum for the elastic scattering of a π-meson on the « a » particle at the total c.m. energy equal to w.

In what follows we will use the formula (1) also for physical values of Δ^2. Such procedure implies the following approximations:

— Use of the vertex function γ_5 instead of the renormalized one $\Gamma_5(\Delta^2)$.

— Use of the propagator $(\Delta^2+\mu^2)^{-1}$ for the intermediate meson instead of $\Delta_{Fc}(\Delta^2)$.

— Substitution of a term proportional to the π-a cross-section to the square of the matrix element describing the upper vertex of our diagrams.

As we have already said, formula (1) holds exactly only in the point $\Delta^2 = -\mu^2$, where the intermediate pion is on the mass shell. However we note that also in the first part of the physical zone it is not very far from being real, due to the fact that the point $\Delta^2 = -\mu^2$ is very near to the physical zone itself. Thus we hope that formula (1) constitutes a good approximation to the true matrix element at least in the region where we will use it $-\Delta^2 < 0.3\,m^2$— where m is the nucleon mass.

In any case the above approximations have exactly the same theoretical justification as the neglecting of diagrams with more than one exchanged pion (*). Indeed following the current philosophy about the location of the singularities of the over-all matrix element in the Δ^2 complex plane (4), we can argue that they are confined on the negative real axis, a pole beeing in the point $\Delta^2 = -\mu^2$ and other singularities being in the range $-4\mu^2 \text{——} -\infty$ for nucleon-nucleon interaction and in the range $-9\mu^2 \text{——} -\infty$ for pion-nucleon interaction, due to the impossibility of two pion exchanges in this last case, because such exchanges would violate the invariance under the Lee and Yang G parity (9). Thus it is possible to write down a spectral representation of the over-all matrix element of the kind

$$M(\Delta^2, w^2) = \frac{R(w)}{\Delta^2+\mu^2} + \int_{\alpha}^{\infty} \frac{f(\Delta^2)}{\Delta^2+\beta}\,\mathrm{d}\Delta^2, \tag{2}$$

where $\alpha = 4\mu^2$ in the case of nucleon-nucleon and $\alpha = 9\mu^2$ in the case of

(*) We are very indebted to Prof. S. Fubini for this argument.

(9) T. D. Lee and C. N. Yang: *Nuovo Cimento*, **3**, 749 (1956).

pion-nucleon interaction; β is a quantity bigger than, or equal to, α. $R(w)$ is the residue of the pole and can be obtained by comparison with (1). Our approximations consist in neglecting the integral in the right hand side of the equation (2) (*). Such an integral contains both contributions coming from more than one pion exchange diagrams and contributions coming from self-energy and vertex correction effects to the diagrams of Fig. 1. In this way all our approximations appear on the very same footing.

3. – Pion production in proton-proton collisions.

We want now to analyse the quoted experiment by BATSON *et al.* [1] on the proton-proton interaction at 970 MeV with the assumption that the diagram of Fig. 1*a* is the only one important for single pion production. In Section 2 we have tried and justified that a not too bad approximation to the corresponding matrix element should be given by formula (1). Thus in the following analysis we will use (1), the « a »particle being in the present case a proton. Upon integration on w^2 we have

$$\frac{\partial\sigma}{\partial\Delta^2} \simeq \frac{f^2 m}{2\pi q_{\mathrm{LL}}^2 \mu^2} \frac{\Delta^2}{(\Delta^2+\mu^2)^2} \bar{\sigma}_{\pi\mathrm{p}}(\Delta^2) \int_{(m+\mu)^2}^{g(\Delta^2)} q'_\pi(w)\,\mathrm{d}w^2 , \tag{3}$$

where

$$\bar{\sigma}_{\pi\mathrm{p}}(\Delta^2) = \frac{\int_{(m+\mu)^2}^{g(\Delta^2)} q'_\pi(w)\,\sigma_{\pi\mathrm{p}}(w)\,\mathrm{d}w^2}{\int_{(m+\mu)^2}^{g(\Delta^2)} q'_\pi(w)\,\mathrm{d}w^2} , \tag{4}$$

and $(w^{\max})^2 = g(\Delta^2)$ is the upper limit of the region kinematically allowed in the Δ^2, w^2 plane (see [1]). The explicit separation of $\sigma_{\pi\mathrm{p}}$ from the remaining integral is done to have an easier parallel with the problem treated in the next section. The Δ^2 dependence of function (4) is shown in Fig. 2 (**) for an isotopic spin $\frac{3}{2}$ of the pion-nucleon system. Notwithstanding the strong w dependence of $\sigma_{\frac{3}{2}}$, $\bar{\sigma}_{\frac{3}{2}}(\Delta^2)$ is seen to be not very strongly Δ^2 dependent.

Function (3) is plotted in Fig. 3 against T — lab. kin. energy of the recoil

(*) It must be remarked that the first term in the right hand side of (2) is not exactly the quantity we considered in (1), a factor Δ^2/μ^2 beeing lacking. We note however that (2) is exact only for scalar particles and that the true form of (1) is obtained keeping into account the spinorial nature of the nucleon.

(**) In Fig. 2, $\sigma_{\frac{3}{2}}$ is plotted against T, the lab. kin. energy of the recoil nucleon, which is essentially the same as to use Δ^2, owing to the well known relation $\Delta^2 = 2mT$.

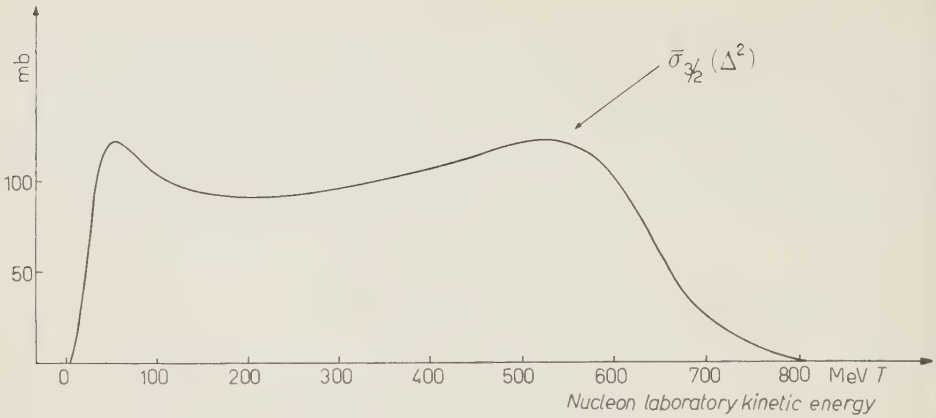

Fig. 2.

nucleon — for a total energy of the two colliding protons in their c.m. of 2324 MeV corresponding to the lab. kin. energy of 970 MeV, taking $f^2 = 0.16$ and $\sigma_{\pi N} = \sigma_{\frac{3}{2}}$, *i.e.* for a π^+ emitted in the « lower » vertex.

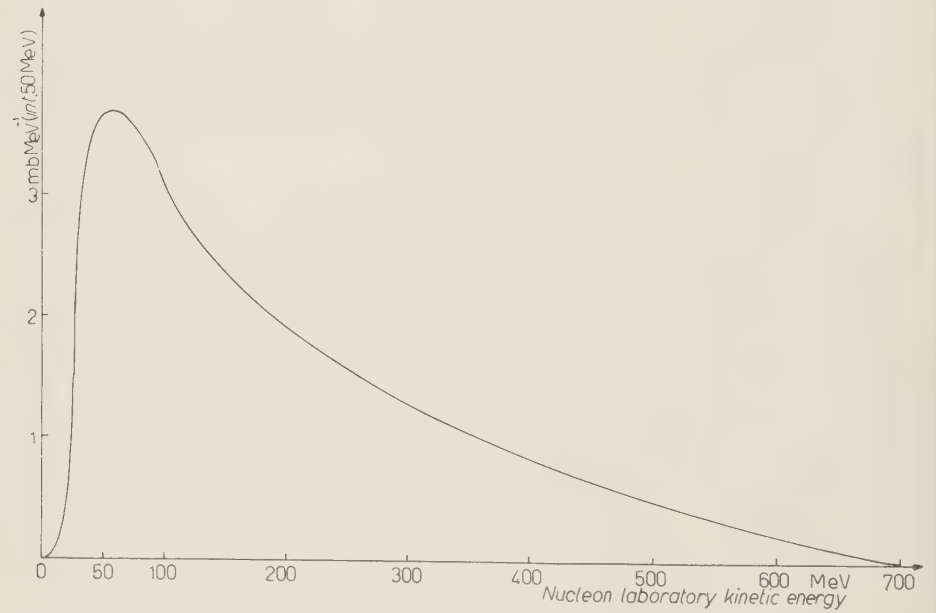

Fig. 3.

The most striking feature is a strong maximum at $T \simeq 60$ MeV.

In Fig. 4 and 5 are shown the experimental spectra of about 270 events of the kind $p+p \to p+n+\pi^+$ [10] with respect to the variables T_p and T_n

[10] L. RIDDIFORD: private communication.

in comparison with the spectrum predicted by the statistical theory, normalized to the same area. The disagreement is evident and is essentially due to a strong maximum for low values of T_p and T_n, about in the same positions

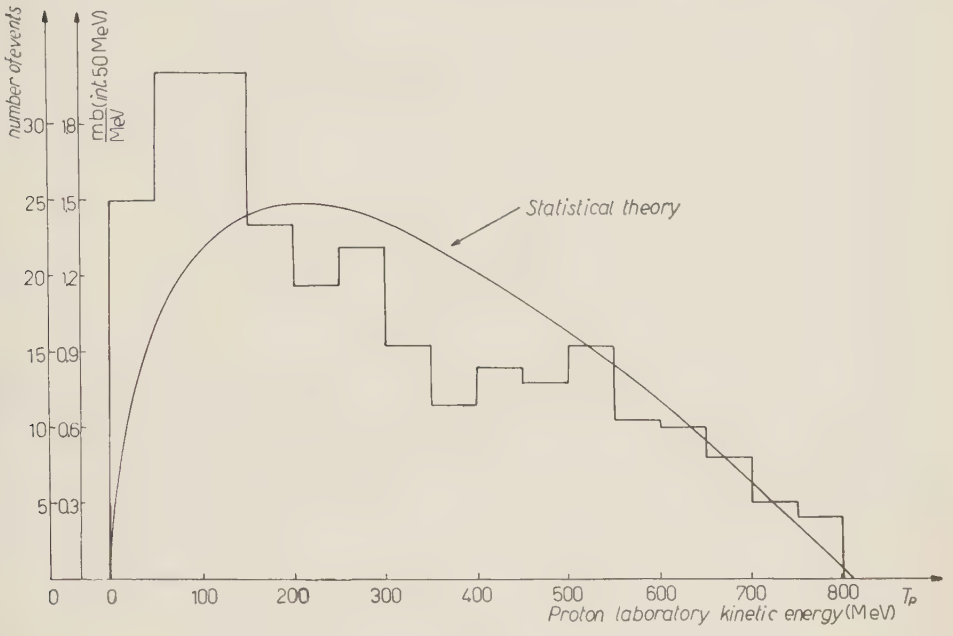

Fig. 4.

predicted by (3). However the highness of the predicted maximum by formula (3) is bigger by a factor 1.8 in the case of the neutrons and lower by a factor 0.2 in the case of the protons. We want now to make clear that also in the case that our assumption that the only important diagrams are those with only one intermediate pion is exactly true, an exact agreement between experiments and the simple formula (3) is not to be expected. Indeed for our events there are two diagrams of the kind shown in Fig. 1*a* that contribute, the first one corresponding to the emission of a π^0 from the « lower » proton that propagates and interacts with change-exchange with the « upper » proton, the second one to a π^+ emitted and interacting elastically with the upper proton. Thus, roughly speaking, the observed neutrons come sometimes from the first diagram where they are something like decay neutrons of an isobar 33, sometimes from the second where they are recoil neutrons. Therefore their spectrum must be given by the superposition, with interference, of the spectra predicted by the two diagrams for the recoil and decay neutrons respectively. Similar reasoning holds obviously also for the protons.

In other words, agreement with (3), that is to say considering the nucleons as coming always from the « lower » vertex of Fig. 1*a*, is *a priori* not to be

expected. We are not able to imagine other matrix elements with such a strong variability in the low Δ^2 region as to have a maximum similar to that predicted by (3), due to the fact that other matrix elements have singularities

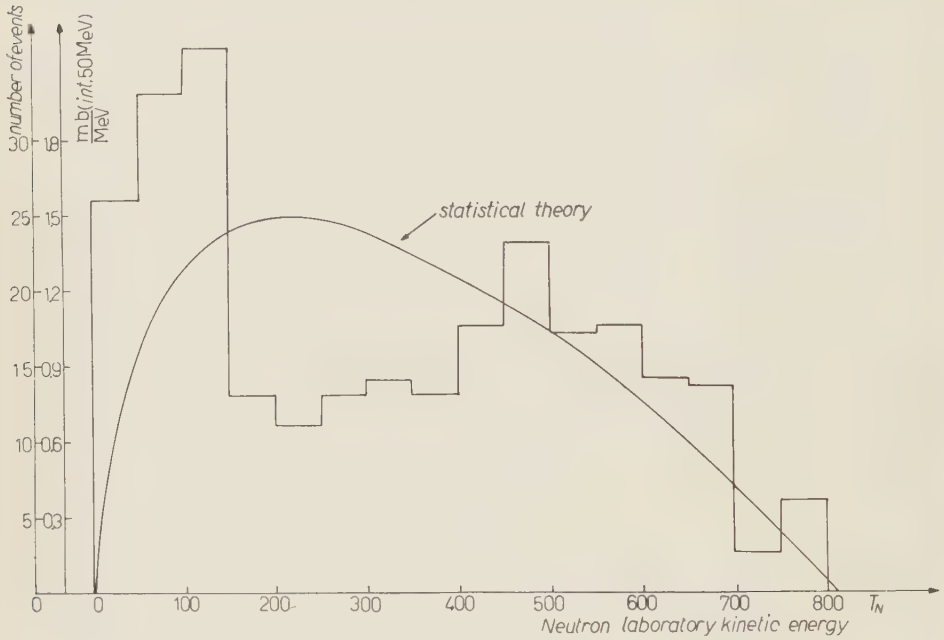

Fig. 5.

more far in the complex Δ^2 plane. Thus we believe that the most plausible conclusion that can be obtained from Fig. 3 and 4, 5 is that the one intermediate pion diagrams give an important contribution to the events we are examining and that it is not excluded that they give the only important contribution.

4. – Pion production in pion-proton collisions.

Let us now consider the quoted experiment (2) on the pion production in π^-p collisions at 960 MeV.

The measurable single pion production processes in π^-p collisions are:

I) $$\pi^- + \mathrm{p} \to \pi^- + \pi^+ + \mathrm{n}\,,$$

II) $$\pi^- + \mathrm{p} \to \pi^- + \pi^0 + \mathrm{p}\,.$$

To each of these processes there is only one diagram of the kind designed in Fig. 1*b* which can contribute.

In the case of I) (II)) the upper vertex is describable as a $\pi^-\pi^+$ ($\pi^-\pi^0$) elastic scattering process.

Formula (1) written down for the present case after integration upon w^2 reads:

$$\frac{\partial\sigma}{\partial\Delta^2} \simeq \frac{\alpha f^2}{2\pi q^2_{\pi L}\mu}\,\frac{\Delta^2}{(\Delta^2+\mu^2)^2}\,\bar{\sigma}_{\pi\pi}(\Delta^2)\int_{(2\mu)^2}^{(\Delta^2)} q'_\pi(w)\,\mathrm{d}w^2\,, \tag{5}$$

where

$$\bar{\sigma}_{\pi\pi}(\Delta^2) = \frac{\int_{(2\mu)^2}^{g(\Delta^2)} \sigma_{\pi\pi}(w)\,q'_\pi(w)\,\mathrm{d}w^2}{\int_{(2\mu)^2}^{g(\Delta^2)} q'_\pi(w)\,\mathrm{d}w^2}\,, \tag{6}$$

and $\alpha = 2$ (1) in the case of events I) (II)). If $\sigma_{\pi\pi}$ is not very varying $\bar{\sigma}_{\pi\pi}$ gives its mean value and is almost independent from Δ^2. If $\sigma_{\pi\pi}$ has instead resonance-like peaks $\bar{\sigma}_{\pi\pi}$ is expected to have a not very strong but yet observable Δ^2 dependence. To have an idea one can look to Fig. 2 where the similar $\bar{\sigma}_{\pi p}$ is drawn for the case of total iso-spin $\frac{3}{2}$ where the well-known 33 resonance is present.

In Fig. 6 is shown the quantity

$$\frac{1}{\bar{\sigma}_{\pi\pi}}\frac{\partial\sigma}{\partial T}\,,$$

as calculated with formula (4) for the experiment under investigation. Also in this case we have a pronounced peak for low T ($T \simeq 50$ MeV). Multiplication by $\bar{\sigma}_{\pi\pi}$ could somewhat shift the position of this maximum if $\sigma_{\pi\pi}(w)$ has a resonance.

Now let us discuss the experimental data. We have seen in the preceding section that the neglecting of more than one pion exchanges between the two nucleons is not a bad approximation, at least in the low Δ^2 region. We think, therefore, that the same should be true also in this case *a fortiori*, due to the fact that two pions exchanges are forbidden, together with any even number of pions exchanges. In this case, however, there could be also contributions from the direct interaction of the incoming pion with the nucleon core, but we will assume that these are unimportant. Another important difference between the present case and the one treated in the preceding section is that the diagram of Fig. 1*b* does not contain any final state 33 interaction. We do know however that such interaction is important [2] and thus we are led to the *a priori* conclusion that our diagram cannot be the only one important.

However we will analyse the experimental data with this model in mind:

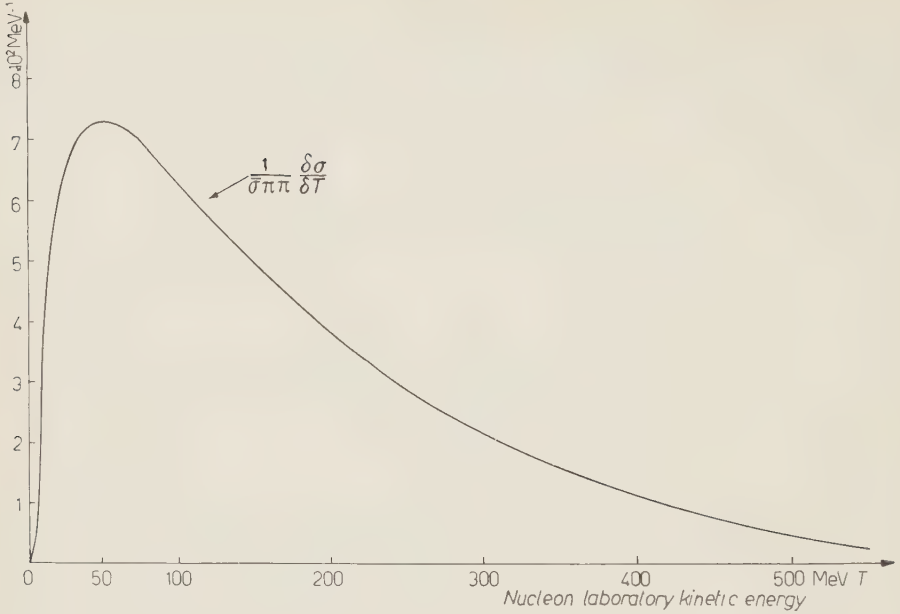

Fig. 6.

the incoming pion hits on a single pion of the cloud and makes it real. After this interaction has taken place the two pions are yet in the volume of interaction and have a certain probability of interacting with the nucleon. As

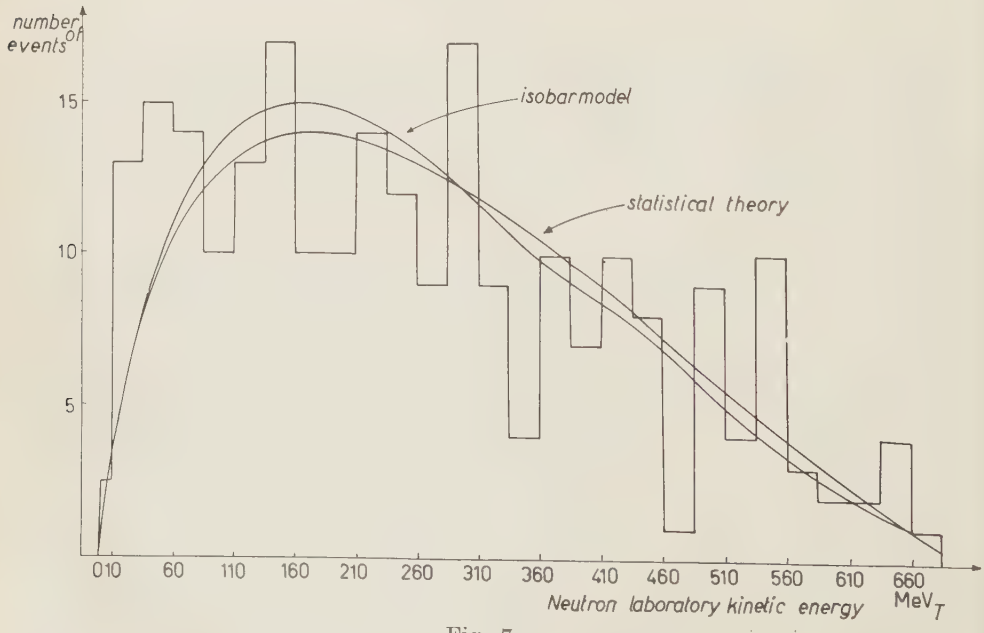

Fig. 7.

suggested by the isobar model one can argue that this probability is measured by the cross-section for pion-nucleon scattering.

For the reaction I) there is the π^-n system that is in a pure $I = \frac{3}{2}$ state. No such system exists in the case of reaction II). This does not mean obviously that the final state interactions are absent for reaction II), but only that they should be less important. Thus we expect that the predicted peak by (5) should be more easily seen for reaction II).

This fact is confirmed by experiments. Experimental spectra as a function of T are shown in Fig. 7 and 8, in comparison with isobar model and statis-

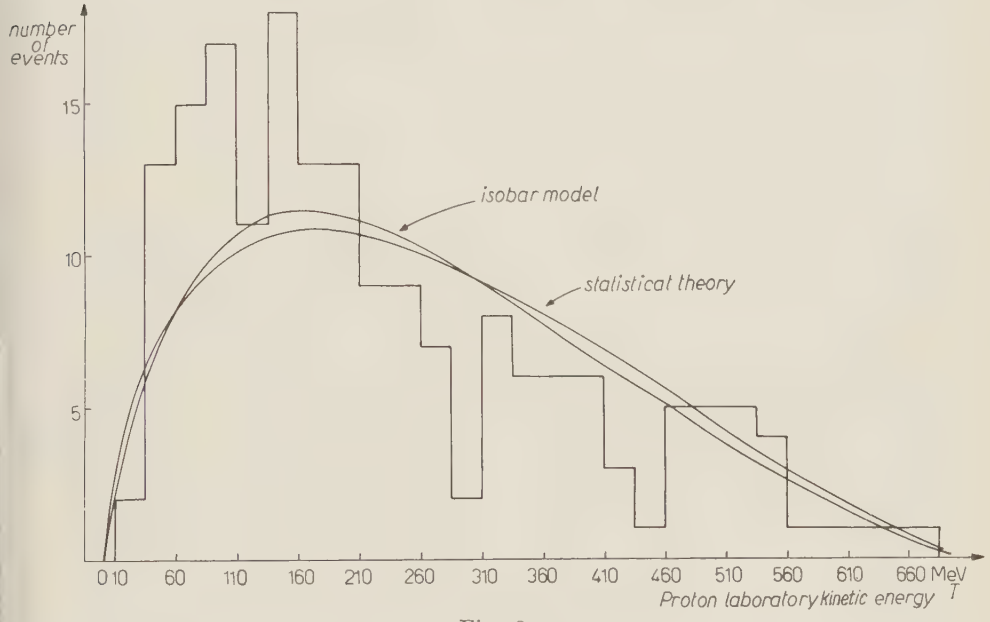

Fig. 8.

tical theory. While in the case of reaction I) the agreement with the isobar model is not too bad, in the case of reaction II) a peak in the low energy region is clearly present. To be precise also in the case of reaction I) there seems to be a peak in the low energy region, but the disagreement from both isobar model and statistical theory is of about two standard deviations. Thus no precise conclusion can be reached in this case on this simple basis. However to investigate the nature of this uncertain peak we have selected the events with lab. kin. energy of the final neutron less than 85 MeV (44 events at all) and constructed their spectrum against the c.m. π^- momentum. This is shown in Fig. 9. There is no trace of the very pronounced peak which is present if the same spectrum is constructed with all events together (see [2]) and which is due to the very frequent formation of the (π^-n) isobar. To give a more

quantitative meaning to this difference we have calculated the ratio R between the number of events with π^+ momentum bigger than 300 MeV/c and

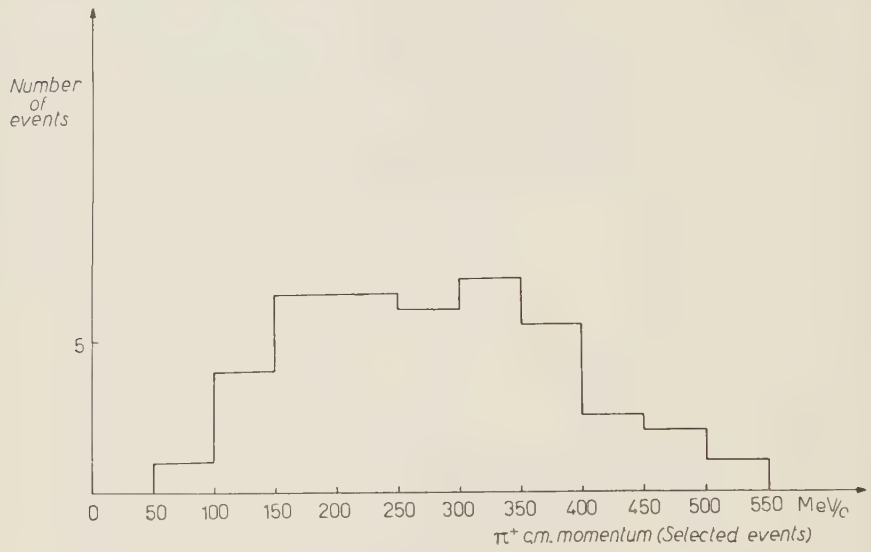

Fig. 9.

the similar number with π^+ momentum lower than 300 MeV/c. It turns out to be

$$R = 2.6 \pm 0.2 \qquad \text{all events together}$$

$$R = 0.75 \pm 0.33 \qquad \text{only selected events}$$

where the indicated errors are the statistical ones.

Explicit calculations have been carried out and it has been found that the selection of low T events does not imply any kinematical restriction on the c.m. momenta of the pions. In other words isobar model predictions for these events are almost the same as for all events together. Thus the preceding quantitative disagreement must be considered as an indication that other matrix elements but those containing final state pion-nucleon interaction are important in the low Δ^2 region. We believe that the most natural interpretation is that the important one is the one drawn in Fig. 1*b*, due to the fact that it was *a priori* expected theoretically to be peaked exactly in this region.

We want to emphasize that this low Δ^2 peak is strongly characterized by the one pion exchange character of the calculated matrix element. Indeed it results from the multiplication of $\Delta^2(\Delta^2+\mu^2)^{-1}$ by the integral in the right hand side of (5). While the latter is something like a phase space factor that should be present in every integrated matrix element, the former is peculiarly

characterized by the one pion exchange and rather strongly Δ^2 dependent. Indeed it is 0 at $T=0$ increases up to a maximum at $T=10$ MeV and then rapidly falls off. As a result of the multiplication of these two the maximum of the phase space factor is shifted in a considerable way toward the low T region. No other matrix element is expected to have such a strong Δ^2 dependence, due to the fact that singularities in the Δ^2 plane are either absent, or much more far than the pole in $\Delta^2=-\mu^2$. Therefore we believe that the most sure conclusion of our paper is a qualitative evidence of a pion-pion interaction.

To try and get something more than a pure qualitative evidence we have fitted the highness of experimental peaks with equation (5) using $\bar{\sigma}_{\pi\pi}$ as an adjustable parameter. The resulting values were

$$\bar{\sigma}(\pi^+ + \pi^- \to \pi^+ + \pi^-) \simeq 20 \text{ mb},$$

$$\bar{\sigma}(\pi^0 + \pi^- \to \pi^0 + \pi^-) \simeq 40 \text{ mb}.$$

We want to stress, however, that such numbers are to be taken just as an indication of the order of magnitude of the pion-pion cross-sections. Indeed we are not able neither to calculate the contributions coming from diagrams with final state pion-nucleon interaction, nor to predict the sign of the interference terms. On the other hand the pion-pion interaction diagram, if our interpretation is correct, should be in any case about as important as other diagrams are. Thus we do not believe that the given cross-sections can be wrong by more of a factor of ten.

Finally, following the ideas outlined in this section, we want to propose an experiment from which more detailed information about $\pi\pi$ interaction should be obtained. It consists of a simple one pion production experiment in pion-nucleon collision, but at an energy sufficiently high as to have a reasonable number of events with Q values of both pions with respect to the nucleon ranging in the interval $(300 \div 500)$ MeV. Selection of such events should give a sample for which final state interactions between the pions and the nucleon should be much less important as in the experiment examined in this section. Thus for these events we expect the low Δ^2 peak to be well detectable.

In the optimistic case that the agreement of the experimental $\partial\sigma/\partial\Delta^2$ with the one given by (5) should be rather good, one could go further and divide it by

$$\frac{\Delta^2}{(\Delta^2+\mu^2)^2}\int_{(2\mu)^2}^{g(\Delta^2)} q'_\pi(w)\,\mathrm{d}w^2 .$$

The so obtained distribution should be constant or again peaked in the

low Δ^2 region according to whether the pion-pion cross-section is constant or strongly peaked. In any case the experiment should be a test of the proposed idea that the only important elementary interaction in single pion production comes from diagrams with pion-pion interaction vertices.

* * *

We are grateful to the components of the Bologna group for having put at our disposition all their experimental results before publication.

We are indebted also with Prof. L. RIDDIFORD for having communicated the unpublished spectra discussed in Section 2 and to the Profs. S. FUBINI, A. MINGUZZI G. PUPPI and A. STANGHELLINI for useful comments on this work.

Finally, we want to acknowledge the precious aid of Mister W. KLEIN and Miss C. PETIT in the numerical computations.

RIASSUNTO

Si propone un semplice modello per lo studio della produzione di un solo pione nell'urto tra una generica particella « a » e un nucleone. Il modello consiste essenzialmente nel trascurare tutte le interazioni eccetto quella tra la particella incidente e un singolo pione virtuale del nucleone. Un'applicazione a un recente esperimento del gruppo di Birmingham (1) fornisce una prova in favore dell'ipotesi. Un'altra applicazione all'esperimento del gruppo di Bologna (2) fornisce la prova qualitativa per l'interazione pione-pione. I dati sperimentali suggeriscono una sezione d'urto pione-pione dell'ordine di 10 mb. Infine si propone un esperimento che dovrebbe fornire informazioni più dettagliate circa l'interazione pione-pione.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori).

A Note on Quasi-Elastic Photon-Deuteron Scattering (*).

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(ricevuto il 27 Aprile 1959)

The usual way of treating quasi-elastic photon deuteron scattering is with the assumption of both the closure and impulse approximations (1). In addition, of course, the elementary scattering of photons from neutrons and protons is not very well known at higher energies so that even within this framework not enough information is available to proceed without further approximations. We have examined the question of corrections to the impulse approximation in the energy region sufficiently above the deuteron binding energy not to be limited to the deuteron Thomson cross section, and fairly below the meson threshold, where the elementary scattering is known, and where it turns out that the calculation is relatively simple.

It is known that at very low photon energy the scattering approaches the deuteron Thomson limit (2). The scattering amplitude used in the impulse approximation, however, approaches the Thomson limit for the two free non-interacting nucleons. Within the static model it appears difficult to achieve the correct low energy limit, except by requiring this limit as in a dispersion relation subtraction, a difficult procedure for quasi-elastic processes. We will ignore this problem and use simple perturbation theory, allowing a single meson to be exchanged by the nucleons as a correction to the impulse approximation where no mesons are exchanged. The scattering amplitude so obtained approaches neither the deuteron Thomson limit, nor the proton Thomson limit, but even a larger value at low energy.

(*) This work is supported in part through AEC Contract AT(30-1)2098, by funds provided by the U.S. Atomic Energy Commission, the Office of Naval Research and the Air Force Office of Scientific Research.

(1) R. H. Capps: *Phys. Rev.*, **106**, 1031 (1957). This paper includes references to earlier papers.

(2) F. E. Low: *Phys. Rev.*, **96**, 1428 (1955); M. Gell-Mann and M. L. Goldberger: *Phys. Rev.* **96**, 1433 (1955).

The differential cross section with polarization averaged over assumes the form:

$$d\sigma_d = \left(\frac{1}{4\pi}\right)^2 \cdot \frac{1}{2} \cdot (1 + \cos^2\theta) \left[\left(\frac{e^2}{M}\right)^2 + \frac{1}{2}\int \frac{d^3q}{(2\pi^3)} \int d^3x \varphi^2(x) \frac{e^2}{2M} \cdot \right.$$

$$\left. \cdot 4\pi \cdot \frac{4}{3} e^2 f_\tau^2 \left[\frac{2 \exp[i(q-k)\cdot x]}{k^2 - \omega^2}\right] \cdot \left(1 - \frac{4}{3} \frac{q^2}{(q^2+u^2)^2} - \frac{q^2}{q^2+u^2}\right)\right].$$

$\omega^2 = q^2 + \mu^2$; . . . θ = scattering angle;

μ = meson mass; $\varphi(x)$ = deuteron wave function.

The γ-nucleon scattering has been taken as pure Thomson. Certain terms of order (k/μ) have been neglected, and a purely $l=0$ wave function is assumed. Within these assumption the angular dependence is the same as that of the Thomson amplitude. If one includes terms of order $(k/\mu)^2$, using the Hulthén wave function, the result is an addition of about 15% to the Thomson cross section, as well as a change in the angular distribution.

Preliminary results of γ-deuteron scattering in the energy range of 90 MeV seem to indicate deviations from the impulse approximation larger than that obtained by the present calculation [3].

[3] D. H. Frisch: private communication.

Une tentative d'éliminer le paradoxe de Landau.

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(ricevuto il 12 Ottobre 1959)

Ainsi que Landau l'a montré, la charge renormalisée de l'électron a comme expression [1]

$$e^2 \qquad \frac{e_1^2}{1 + (ve_1^2/3\pi)\ln(\Lambda^2/m^2)}, \tag{1}$$

e_1 étant la charge non renormalisée et Λ une limite supérieure d'intégration (« cut-off ») dans l'éspace des moments. Si l'on prend $\Lambda \to \infty$, alors même pour e_1 infiniment grande, l'on obtient une constante de couplage renormalisée nulle et c'est justement ce qui constitue le paradoxe de Landau.

De (1) il s'ensuit

$$Z_3 - 1 \qquad \frac{ve^2}{3\pi}\ln\frac{\Lambda^2}{m^2}, \tag{2}$$

où l'on a noté comme d'habitude $Z_3 = e^2/e_1^2$. Cette relation peut être écrite aussi de la manière suivante.

$$\frac{\Lambda^2}{m^2} = \exp\left|\frac{3\pi}{ve^2}y\right| \tag{3}$$

où $y = 1 - Z_3$. On s'aperçoit facilement que Λ^2, comme fonction de la variable complexe y, est invariante par rapport au groupe

$$Z = y - n\,2\pi i\,\frac{ve^2}{3\pi}, \tag{4}$$

n étant un nombre entier positif ou negatif. Le domaine de discontinuité de ce groupe est une bande parallèle à l'axe réel du plan y (Fig. 1) ayant la largeur $8\pi vg/3$ où $g = e^2/4\pi$ est la constante de couplage du champ électro-magnétique au champ de Dirac.

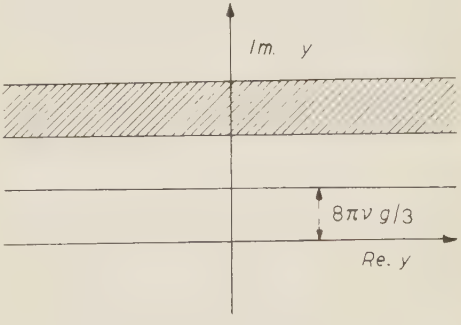

Fig. 1.

Landau lui-même a fait remarquer que le modèle examiné par lui est incomplet, parce qu'il ne considère que

[1] L. D. Landau: *On the Quantum Theory of Fields*, in: *Niels Bohr and the Development of Physics*. W. Pauli, editor.

l'interaction entre l'électron et le photon tandis qu'en realité il peut exister aussi d'autres particules, sauf l'électron, qui contribuent à la polarisation du vide.

Nous allons montrer que si l'on tient compte aussi des autres interactions possibles, le paradoxe de Landau est éliminé. Notre hypothèse fondamentale est que *la nature de l'interaction est déterminée par la forme du domaine de discontinuité du groupe qui laisse invariante la fonction* $A^2(y)$. En choisissant, d'une manière adéquate, le domaine de discontinuité, l'on peut éliminer le paradoxe de Landau. C'est ce que nous voulons montrer par un exemple.

Nous avons demontré plus haut que l'interaction électromagnetique est caracterisée par l'invariance de la fonction $A^2(y)$ par rapport au groupe discontinu

$$Z = y - nikg\,, \tag{6}$$

k étant une constante (*).

En ce qui concerne l'interaction forte, divers auteurs (Pauli, Dancoff, Serber) ont essayé de remplacer les développements en série suivant les puissances entières ascendentes de g par des développements en série suivant les puissances de $1/g$.

En partant de cette idée, nous introduirons une seconde interaction que nous nommerons « forte » et qui sera caractérisée par l'invariance du « cut-off » par rapport au groupe

$$\frac{1}{Z} = \frac{1}{y} + \frac{m}{ikg} \qquad (m \text{ entier}). \tag{7}$$

Si l'on prend $n=m=1$ et l'on introduit les variables t, τ définies par

$$Z = \frac{kg}{i}\tau\,, \quad y = \frac{kg}{i}t\,, \tag{8}$$

(*) Ainsi $A^2(y)$ est une fonction automorphe par rapport à ce groupe.

les relations (6), (7), deviennent

$$\tau = t + 1\,, \tag{9}$$

$$(T),$$

$$\frac{1}{\tau} = \frac{1}{t} - 1\,. \tag{10}$$

$$(U).$$

D'autre part on peut montrer facilement que $U = STS^{-1}$ où la substitution S est definie par

$$\tau = -\frac{1}{t}\,, \tag{11}$$

$$(S),$$

Par conséquent, les deux substitutions (9), (10) peuvent être remplacées par (9), (11). Mais ces deux transformations sont justement les substitutions génératrices du groupe modulaire

$$\tau = \frac{at+b}{ct+d}\,, \qquad ad - bc = 1, \tag{12}$$

où a, b, c, d, sont des nombres entiers (2).

En se rappelant que la fonction modulaire $J(t)$ est une fonction automorphe par rapport au groupe (12), on peut prendre $A^2 = f(J(t))$ ou tout simplement.

$$A^2 = \text{const}\, J(t) = \text{const}\, \frac{g_2^3}{g_2^3 - 27\, g_3^2}\,, \tag{13}$$

$$g_2 = g_2(1,t) = 60 \sum{}' \frac{1}{(m + m't)^4}\,, \tag{13a}$$

$$g_3 = g_3(1,t) = 140 \sum{}' \frac{1}{(m + m't)^6}\,. \tag{13b}$$

Le domaine de discontinuité est, cette fois-ci, l'un des domaines de la Fig. 2, par exemple celui ayant les sommets (*) O, σ, $\bar{\varrho}$ (hachuré).

(2) R. Fricke: *Lehrbuch der Algebra*, Band 2 p. 77.

(*) $\sigma = \frac{8\pi v}{3} g \qquad \varrho = \frac{1}{2}(1 + i\sqrt{3})\frac{8\pi v}{3} g,$

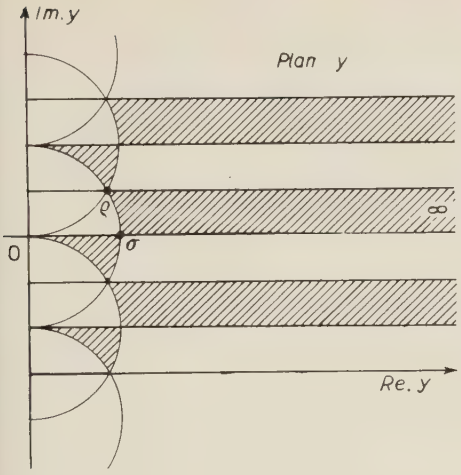

Fig. 2.

La fonction $\Lambda^2(y)$ a un pôle dans le point $y=0$. Par conséquent, pour $\Lambda^2 \to \infty$, $y \to 0$ donc $Z_3 \to 1$. Ainsi, dans ce modèle, le paradoxe de Landau est éliminé.

L'axe imaginaire du plan y est une ligne de points singuliers essentiels de la fonction $\Lambda^2(y)$ d'où il s'ensuit que cette fonction ne peut être prolongée analytiquement dans le demi-plan gauche. Ainsi les valeurs de $\mathrm{Re}\, Z_3 = 1 - \mathrm{Re}\, y$ plus grandes que 1 sont exclues.

Remarquons que si l'on prend le domaine $(\infty, \sigma, \varrho)$ comme « fondamental », $\mathrm{Re}\, Z_3$ peut prendre toutes les valeurs réelles comprises entre $1 - \sigma$ et $-\infty$. Pour $\mathrm{Re}\, Z_3 < 0$, $\mathrm{Im}\, Z_3 = 0$, des états fantômes font leur apparition.

Cet inconvénient n'existe pas dans le cas du domaine $(O, \sigma, \bar{\varrho})$ parceque dans ce cas $1 - \sigma \leqslant \mathrm{Re}\, Z_3 \leqslant 1$, $(1 - \sigma > 0)$.

On peut donc éliminer les fantômes en définissant d'une manière convenable le domaine fondamental.

A Possible Explanation for the Multi-Nucleon K^- Capture.

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(ricevuto il 9 Novembre 1959)

Recently it has been shown [1,2] that the multi-nucleon K^- capture in heavy nuclei plays an important role and that about 35% of all K^- captures at rest in nuclear emulsions [2,3] proceed via the multinucleon mode. The relative frequency of this mode was determined by a detailed analysis of K^- captures at rest in nuclear plates. This determination was based essentially upon the observation of fast baryons, which could be emitted only if the K^- was absorbed by two nucleons simultaneously (with no real pion production) via the reaction: $K^- + 2\mathcal{N} \to Y + \mathcal{N}$. The best estimate of the absolute yield of the $2\mathcal{N}$-captures was obtained [2], by assuming charge independence, from the yield of the single nucleon (pion producing) K^- captures [2,4], namely $K^- + \mathcal{N} \to Y + \pi$.

At the same time it was pointed out theoretically [5] that the K^- absorption in complex nuclei is expected to take place far outside in the periphery of the nucleus. This agrees rather well with the existing experimental data [1,2]. Thus it becomes very difficult to explain the large fraction of the two nucleon mode in nuclear emulsions.

Wilkinson [6] has suggested an explanation for the above discrepancy by assuming the presence of substantial α-particle clusters far outside the core of the nucleus.

In the present note we wish to suggest an entirely different explanation for the relatively high yield of the multinucleon capture in complex nuclei, as well as some other features of this process.

Our basic idea is that the multinucleon reaction is a two-step process. In the first step the K^- is absorbed by a single nucleon forming a resonant K^--$\mathcal{N}$ state (to be called here Σ^*) with sufficiently long life-time for its wavefunction to spread over a substantial part of the nucleus, and in the second

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(1) European Collaboration Data, M. F. Kaplon's Report, *Proc. of CERN Conf.* (1958).

(2) Y. Eisenberg, W. Koch, M. Nicolić M. Schneeberger and H. Winzeler: *Nuovo Cimento*, **11**, 351 (1959), and preprints V and VI. Bern Emulsion Group (1959).

(3) European Collaboration Data, reported by E. H. S. Burhop at the *Kiev Conference*, (July 1959).

(4) European Collaboration: *Nuovo Cimento*, **12**, 91 (1959).

(5) P. B. Jones: *Phil. Mag.*, **3**, 33 (1958).

(6) R. D. Wilkinson: *Phil. Mag.*, **4**, 215 (1959).

step the Σ^* interacts with another nucleon giving a two-baryon final state, namely: $\Sigma^* + \mathcal{N} \to Y + \mathcal{N}$. The existence of a similar resonant state was suggested by MATTHEWS and SALAM (7), in order to account for the K^--p scattering data at low energies.

The life-time τ of the resonant state is given by

$$[(\Gamma/2)^2 + (E - E_0)^2]^{-\frac{1}{2}} .$$

If the Σ^* shouldt raverse ~ 10 fermi before decaying ($\tau \sim (1 \div 2) \cdot 10^{-22}$ s), a total width Γ of ~ 10 MeV is required. A similar width seems to exist for the peak at about $E_0 = 25$ MeV K^- kinetic energy, in the K^--p scattering data (8,9). The Σ^* life-time at 10 MeV off the center of the resonance is reduced only by a factor of 2. Thus it still has a good chance of making the reaction $\Sigma^* + \mathcal{N} \to \to Y + \mathcal{N}$ and about 35% of multi-nucleon captures is then not surprising.

The Q-value of the Σ^* should be divided between the two baryons and this agrees with the experimentally observed energy-distribution of the $2\mathcal{N}$-interaction products (1-3). We thus see that an absorption of a K^--meson at the periphery of the nucleus may lead to multi-nucleon capture, provided we adopt the idea of the formation of a relatively long living Σ^*.

Some additional experimental information regarding the multi-nucleon K^--capture has recently become available:

1) the multi-nucleon capture mode seems to be almost absent (only about 1%) in K^- absorptions in deuterium (9).

2) There is a strong indication (2) that in nuclear emulsions, most of the K^- captures leading to a multi-nucleon process occur on the heavy elements (AgBr) and very little on the light elements (CNO).

It is extremely hard to find an explanation for the above observations, by using the ordinary direct interaction treatment (2). However, within the framework of our model, we may suggest a reasonable explanation: the Σ^* formation in D and in the light nuclei may be inhibited on energy grounds. Σ^* production in the center of the resonance (namely ~ 25 MeV) would require about 25 MeV + binding energy of the proton + binding energy of the K^- in the atom. The required energy (about 35 MeV in heavy nuclei and 27 MeV in deuterium) can only be provided by the binding energy of the Σ^* in the residual nucleus. 27 MeV binding energy for the $\Sigma^*\mathcal{N}$-system in deuterium is highly improbable, therefore the reaction could take place only on the low energy tail of the resonance with a corresponding reduction in frequency. The absolute rate of the $2\mathcal{N}$-production is thus expected to increase with A (for small A) and will probably still be small for He and light emulsion nuclei (CNO).

Finally, we wish to make a few remarks about the multi-nucleon reaction rates. The emulsion (1,2) as well as the bubble-chamber results (9) show clearly that the K^--$\mathcal{N}$ interaction at low energies proceeds mainly via the $T = 0$ channel, and that only at higher relative K-$\mathcal{N}$ momentum the $T = 1$ state becomes important (2,9,10). It is thus reasonable to assume that the K^--$\mathcal{N}$ resonant state occurs for $T = 0$, namely that the Σ^* has a zero isotopic spin and therefore is formed in K^- captures by protons only. The multi-nucleon reaction rates for such a case are given in Table I.

(7) P. T. MATTHEWS and A. SALAM: *Phys. Rev. Lett.*, **2**, 226 (1959).

(8) M. F. KAPLON's Report: *Proc. of the CERN Conference* (1958).

(9) L. W. ALVAREZ' Report on the results of the Berkeley H and D bubble chamber groups: *Kiev Conference* (July 1959).

(10) Y. EISENBERG, W. KOCH, E. LOHRMANN, M. NICOLIĆ, M. SCHNEEBERGER and H. WINZELER: *Nuovo Cimento*, **9**, 745 (1958).

TABLE I. – *Reaction rates from the* Σ^* *model (with* $T=0$).

$K^- + nn \longrightarrow \Sigma^- + n$	0
$K^- + pn \to \Sigma^* + n \to \Sigma^- + p$	$\frac{2}{3}P^2$
$\to \Sigma^0 + n$	$\frac{1}{3}P^2$
$\to \Lambda^0 + n$	Q^2
$K^- + pp \to \Sigma^* + p \to \Sigma^+ + n$	$\frac{2}{3}P^2$
$\to \Sigma^0 + p$	$\frac{1}{3}P^2$
$\to \Lambda^0 + p$	Q^2

$P = \langle \frac{1}{2} | H_\Sigma | \frac{1}{2} \rangle$, $\quad Q = \langle \frac{1}{2} | H_\Lambda | \frac{1}{2} \rangle$

(The reaction rates calculated by the conventional direct treatment were published before [2]). Thus, if the Σ^* has indeed $T=0$, the $2\mathcal{N}$ K^- captures proceeding via the mechanism proposed here should not yield Σ^-n in the final state (see Table I), and the ratio of the reactions Σ^-p/Σ^+n and Λ^0n/Λ^0p should be equal to $n/(p-1)$, namely ~ 1.3 for the heavy emulsion nuclei. The experimental situation concerning the multi-nucleon reaction rates is not very clear yet, but it seems that the frequency of the K^- $2\mathcal{N}$ capture leading to Σ^-n (if it exists at all) is very small compared with the Σ^-p yield [1,3], and that also the ratio Σ^-p/Σ^+n is about 1.5. A rough estimate of the Λ^0n/Λ^0p ratio [2] indicates that it is also not very different from 1.5.

We wish to conclude by saying that the Σ^* model proposed here explains, in a natural way, the observed ratios between the various multi-nucleon reaction rates. It also predicts that this ratios should be energy independent. In addition to that, the Σ^* model offers an explanation for the apparant paradox of the high yield of multi-nucleon processes in heavy nuclei in spite of peripheral K^- absorption and in contrast to the practical non-occurance of a similar process in the light nuclei.

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Canonical Variables, Expression for Energy, and the Criteria for Radiation in General Relativity (*).

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A canonical form for general relativity involving only two pairs of unconstrained conjugate variables has been obtained. We begin with the action

$$I = \int \sqrt{-g}\, R \, \mathrm{d}^4x \,,$$

expressed in first order form (1):

$$(1) \qquad \sqrt{-g}\, R = -g_{ij}\, \partial_t \pi^{ij} + N({}^3g)^{-\frac{1}{2}}({}^3g^3R + \tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}) + \\ + 2\eta_i\, \pi^{ij}{}_{|j} - 2\left\{({}^3g)^{\frac{1}{2}} N_{,i} + (\pi_{ij} - \tfrac{1}{2} g_{ij}\, \pi)\, \eta^j\right\}^{|i} \,,$$

Here

$$\pi^{ij} = \sqrt{-g}\,[\Gamma^0{}_{lm} - g_{lm}\Gamma^0{}_{rq}\gamma^{pq}]\gamma^{li}\gamma^{mj} \,,$$

γ^{ij} is the matrix inverse to g_{ij}; these are used to raise and lower latin indices. Further,

$$\eta_i = g_{0i} \,, \qquad N = (-g^{00})^{-\frac{1}{2}} \,, \qquad {}^3g = \det g_{ij} \,,$$

(*) Supported in part by a National Science Foundation research grant. The detailed version of this work, which consists of two papers, will appear in *Phys. Rev.*, in Vol. **117** or **118** (1960).

(**) Alfred P. Sloan Research Fellow. On leave from Palmer Physical Laboratory, Princeton University, Princeton, N. J.

(1) We use units such that $16\pi\gamma c^{-4} = 1$, $c = 1$. Latin indices run over 1, 2, 3, greek indices over 0, 1, 2, 3, and $f_{,i} = \partial f/\partial x^i$.

${}^3R$ the curvature scalar formed from g_{ij} and the bar notation is used for the covariant derivative with respect to the g_{ij} metric. Variation of this action gives rise to all the usual Einstein field equations in Palatini form. Endpoint variations give the generator

$$G = \int \mathrm{d}^3x [-g_{ij}\delta\pi^{ij} + T'^0_{\ \mu}\,\delta x^\mu] , \tag{2}$$

where the stress tensor terms,

$$\int \mathrm{d}^3x\, T'^0_{\ \mu}\,\delta x^\mu ,$$

vanish as a consequence of the constraint equations

$$G^0_\mu \equiv R^0_\mu - \tfrac{1}{2} R\delta^0_\mu = 0 ,$$

(obtained from varying Eq. (1) with respect to N, η_i).

We now make use of an orthogonal decomposition of g_{ij} and π^{ij} according to the general scheme

$$f_{ij} = f_{ij}^{\ TT} + \tfrac{1}{2}[f^T\delta_{ij} - (1/\nabla^2)f^T_{,ij}] + f_{i,j} + f_{j,i} , \tag{3}$$

where $1/\nabla^2$ is the inverse of the flat space Laplacian operator with appropriate boundary conditions,

$$f^T = f_{ii} - (1/\nabla^2)f_{ij,ij} , \tag{4a}$$

$$f_i = (1/\nabla^2)[f_{ij,j} - (1/2\nabla^2)f_{kj,kji}] , \tag{4b}$$

and the transverse traceless part of f_{ij}, *i.e.*, $f_{ij}^{\ TT}$ ($f_{ij}^{\ TT}{}_{,j}=0$, $f_{ii}^{\ TT}=0$) is the remainder. Such a decomposition is meaningful only when a set of coordinate conditions are imposed. We choose the following:

$$g_{ij,j} = 0 , \qquad \pi^{ij}{}_{,jj} - \pi^{ij}{}_{,ij} = 0 . \tag{5}$$

General covariance is maintained, since imposition of these coordinate conditions is equivalent to using certain invariant functionals of the metric as independent variables in place of coordinates ([2]). These functionals are defined to within Lorentz transformations. Conditions (5) imply that we are choosing the functionals ([3])

$$t = -(1/2\nabla^2)\pi^T , \qquad x^i = 2g_i , \tag{6}$$

as expressed in the frame defined by (5), and ensure that the metric be asymptotically flat.

([2]) Thus, as is well known, the apparently non-covariant De Donder conditions, $(\sqrt{-g}\,g^{\mu\nu})_{,\nu} = 0$ are equivalent to the covariant statement that four linearly independent scalar functions h^α are to be used as the coordinates x^α, where $h^\alpha{}_{;\mu}{}^{;\mu} = 0$. Such an equivalence between coordinate conditions and covariant statements holds for our case and indeed for any other choice of coordinates which does not depend on the x^μ of some initial frame.

([3]) Appropriate boundary conditions have, of course, been imposed on the operator $1/\nabla^2$ in order to relate eqs. (5) and (6). A simple procedure making this operator manifestly non-singular consists in rewriting eq. (6) as $t\exp[-\alpha r] = -(\tfrac{1}{2}\nabla^2)\pi^T$, $x^i\exp[-\alpha r] = 2g_i$ and subsequently taking the limit $\alpha \to 0$ at the end of the analysis.

Inserting the orthogonal decomposition into the generator of Eq. (2), one finds after some partial integrations that

$$G = \int \mathrm{d}^3x \, [\pi^{ijTT}\delta g_{ij}{}^{TT} + g^{T}{}_{,ii}\delta\{(-1/2\nabla^2)\pi^{T}\} - 2\pi^{ij}{}_{,j}\delta g_i] \,, \tag{7}$$

since both the variation δ and the time derivative in the action commute with the decomposition. Here $g^{T}{}_{,ii}$ and $\pi^{ij}{}_{,j}$ may be expressed in terms of $g_{ij}{}^{TT}$ and π^{ijTT}, by solving the four constraint equations [4] $G^0{}_\mu = 0$ for them. We see, therefore, that the generator of Eq. (7) is in standard canonical form $G = p\,\delta q - H\,\delta t$ with the additional momentum terms $T^0{}_i$ characteristic of a field theory:

$$G = \int \mathrm{d}^3x [\pi^{ijTT}\delta g_{ij}{}^{TT} - \mathscr{H}\,\delta t + T^0{}_i\,\delta x^i] \,. \tag{8}$$

Since x^i and t do not appear explicitly in the expressions for g_{ij} and π^{ij} they do not appear explicitly in the equations $G^0{}_\mu = 0$. Detailed calculation shows that they also do not appear in the solutions for $g^{T}{}_{,ii}$ and $\pi^{ij}{}_{,j}$. Thus the quantities

$$\mathscr{H} = -g^{T}{}_{,ii}[g_{ij}{}^{TT}, \pi^{ijTT}] \,, \tag{9a}$$

$$T^0{}_i = -\pi^{ij}{}_{,j}[g_{ij}{}^{TT}, \pi^{ijTT}] \,, \tag{9b}$$

do not depend explicitly on the coordinates [5]. This coordinate independence allows one to derive the standard conservation laws. From Eq. (8) and the fact that the action of Eq. (1) reduces to

$$I = \int \mathrm{d}^4x [\pi^{ijTT}\partial_t g_{ij}{}^{TT} - \mathscr{H}[\pi^{ijTT}, g_{ij}{}^{TT}]] \,. \tag{10}$$

one sees that the two independent pairs of canonical variables are $g_{ij}{}^{TT}$ and π^{ijTT}. These, therefore, obey simple Poisson bracket relations

$$[g_{mn}(\boldsymbol{r})^{TT}, \pi^{ij}(\boldsymbol{r}')^{TT}] = \delta^{ij}{}_{mn}(\boldsymbol{r}, \boldsymbol{r}')^{TT} \,, \tag{11}$$

where the transverse traceless δ-function, $(\delta^{ij}{}_{mn})^{TT}$, is defined as in linearized theory [6] and is, of course, independent of the metric.

The transformation properties of $g_{ij}{}^{TT}$, π^{ijTT} and $\mathscr{H} = -g^{T}{}_{,ii}$ under Lorentz transformations are the same as in linearized theory [6]. Hence, the Lorentz invariance of the canonical action, Eq. (10), of the field theory follows from that of linearized theory. As a consequence, one can in principle derive a symmetric stress tensor obeying the usual law, $T^{\mu\nu}{}_{,\nu} = 0$. Similarly, an angular momentum conser-

[4] That $G^0{}_\mu = 0$ has at least perturbation series solutions for $g^{T}{}_{,ii}$ and $\pi^{ij}{}_{,j}$ can easily be checked, as can the fact that our coordinate conditions are maintained in such an expansion. The Hamiltonian can thus be exhibited as an infinite series in powers of the canonical variables beginning with the linearized theory's Hamiltonian.

[5] The importance of this requirement in general relativity is discussed in a paper to appear in *Phys. Rev.*, **116**, (1959) Dec. 1 issue.

[6] R. Arnowitt and S. Deser: *Phys. Rev.*, **113**, 745 (1959).

vation law holds. $T^{\mu\nu}$ is a Lorentz tensor with respect to the Lorentz transformations relating different choices of independent variables within our coordinate conditions; this is precisely the sense in which $T^{\mu\nu}$ is a Lorentz tensor in special relativistic field theories.

The energy E of the field is defined to be the numerical value of the Hamiltonian for the given solution of the equations of motion. Thus, in the evaluation of

$$E = -\int g^{T}{}_{,ii}\, \mathrm{d}^3x\,, \tag{12}$$

$g^{T}{}_{,ii}$ need not be expressed in terms of the canonical variables and so E can be evaluated as a surface integral (even though the Hamiltonian cannot be reduced to this form). From Eq. (4*a*) we obtain

$$E = -\int g^{T}{}_{,i}\, \mathrm{d}S_i = -\int (g_{jj,\,i} - g_{ij,\,j})\, \mathrm{d}S_i\,, \tag{13}$$

where $\mathrm{d}S_i \equiv \frac{1}{2}\varepsilon_{ijk}\,\mathrm{d}x^j\,\mathrm{d}x^k$ is the two dimensional surface element at infinity in rectangular coordinates. We have assumed that the metric becomes asymptotically flat and the coordinates rectangular at spatial infinity. Coordinate transformations preserving these boundary conditions can there be rigorously treated by the linearized theory where g^T is a coordinate scalar ([6]). Thus, it is not necessary to express the metric in the canonical coordinate system to evaluate the energy; neither is it necessary to use rectangular coordinates provided one makes the usual flat space tensor transformations to the desired (*e.g.* spherical) coordinates. This definition effectively states that the energy of a closed system may be obtained from the coefficient of $1/r$ in the asymptotic expansion of g^T. It is also the gravitational mass of the system as seen by a distant test particle (due to the boundary conditions). The constancy of the energy insures that this coefficient is time independent. Eq. (13) holds without modification when a point particle is coupled to the gravitational field; E now represents the total energy of the coupled system. For the simple case of the Schwarzschild metric one of course finds that the energy is the mass parameter.

It is a basic criterion for the energy and momentum of a closed system that they involve only those quantities required to specify the state of the system at a given time. For general relativity these variables are g_{ij} and π^{ij} when no coordinate conditions are employed ([7,8]), but do not include, for example, $g_{0\mu}$ which serve to describe how the coordinates will be chosen at a later time. Our expression for the energy satisfies this condition. Analogous considerations to the above hold for the total momentum of the gravitational system where again π^i is a coordinate scalar in the asymptotic region. An example of an initial value problem where $g_{0\mu}$ is not determined is the « many body » spatial metric of Lichnerowicz ([9,8])

$$g_{ij} = \delta_{ij}[1 + \Sigma\alpha_n/2\,|\boldsymbol{r} - \boldsymbol{a}_n|]^4\,, \tag{14a}$$

$$\pi^{ij} = 0\,, \tag{14b}$$

Here Eq. (13) yields unambiguously the value $\Sigma\alpha_n$ for the energy.

([7]) Y. FOURES-BRUHAT: *Journ. Rational Mech. Anal.*, **4**, 951 (1956).
([8]) See also C. W. MISNER and J. A. WHEELER: *Ann. Phys.*, **2**, 589 (1957).
([9]) A. LICHNEROWICZ: *Journ. Math. Pure Appl.*, (9) **23**, 37 (1944).

Our formalism also allows us to establish the criteria for the existence of gravitational radiation; these are in exact correspondence to those of electromagnetic theory. Thus, the condition for radiation escaping to infinity is stated in terms of the Poynting vector there. The energy flux across a surface element $\mathrm{d}S_i$ at infinity is ([10])

$$T^0{}_i \, \mathrm{d}S_i = - \pi^{ij}{}_{,j} \, \mathrm{d}S_i \, , \tag{15}$$

where, again the right hand side may be evaluated in any asymptotically rectangular coordinate system. More generally, the criterion for the existence of gravitational « waves » at any point is the presence (in the canonical coordinate frame) of a non-vanishing $g_{ij}{}^{TT}$ or π^{ijTT} there. Alternately, such a situation means that there exists an excitation of the gravitational field in one of its canonical variables (which are independent of the source variables). This is identical to the electromagnetic definition of a wave, which requires the existence of $A_i{}^T$ or $E_i{}^T$, the canonical variables of the electromagnetic field. In the obvious case where no waves are expected to exist, namely the Schwarzschild solution, one can verify that $g_{ij}{}^{TT}$ and π^{ijTT} vanish everywhere in the canonical frame, which justifies its being regarded as a « one-particle » solution.

It is hoped that the reduction to canonical form here obtained can be paralleled in quantum theory.

* * *

We wish to thank the Gudhjem, Bornholm, School Commission for their hospitality in providing us with working space while this investigation was performed.

([10]) In Eq. (15), we have used the momentum density $T^0{}_i$ in place of the Poynting vector T^{i0}. With the stress tensor used here, these may differ from each other by a divergence of canonical variables, since the symmetry requirement was not imposed explicitly in obtaining $T^0{}_i$.

The Influence of the Anomalous Moments of the Baryons in the Decay of π^0 in two γ Rays.

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(ricevuto il 23 Novembre 1959)

Sakata and Tanikawa [1] were the first to notice the possibility of a rapid decay of the π^0-meson in two γ-rays through the virtual creation of a proton-antiproton pair. Their estimation gave — for the lifetime of the π^0 at rest, the value $\tau \sim 10^{-16}$ s. Later, other authors [2] studied the same problem from a more general point of view and recently Kinoshita [3] and Tiomno [4] have considered the inclusion of intermediary pairs of hyperons in the decay of π^0.

Marshak [5] in his book asked about the influence of the anomalous magnetic moments of the intermediary nucleon pairs on the lifetime of $\pi^0 \to 2\gamma$ decay and presumed that it could be small. Our purpose here is to study the influence of the anomalous magnetic moments of the intermediary pairs of baryons.

The interaction hamiltonian density of a fermion field with charge e and anomalous magnetic moment k with an electromagnetic field A_μ is given by:

$$H = -ie\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x) - \frac{ke}{2M}\bar{\psi}(x)\sigma_{\mu\nu}\psi(x)F_{\mu\nu}(x)\,, \tag{1}$$

where M is the mass of the fermion,

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \qquad \text{and} \qquad F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\,.$$

For the interaction between meson π^0 and baryon fields, we shall assume the density:

$$H' = \{ig_{\mathcal{N}\pi}[\bar{p}\gamma_5 p - \bar{n}\gamma_5 n] + ig_{\Lambda\pi}[\bar{\Sigma}^0\gamma_5\Lambda^0] + \\ + ig_{\Sigma\pi}[\bar{\Sigma}^+\gamma_5\Sigma^+ - \bar{\Sigma}^-\gamma_5\Sigma^-] + ig_{\Xi\pi}[\bar{\Xi}^0\gamma_5\Xi^0 - \bar{\Xi}^-\gamma_5\Xi^-]\}\pi^0\,. \tag{2}$$

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(1) S. Sakata and Y. Tanikawa: *Phys. Rev.*, **57**, 548 (1940).

(2) R. J. Finkelstein: *Phys. Rev.*, **72**, 415 (1947); J. Steinberger: *Phys. Rev.*, **76**, 1180 (1949); H. Fukuda and Y. Miyamoto: *Progr. Theor. Phys.*, **4**, 347 (1949).

(3) T. Kinoshita: *Phys. Rev.*, **94**, 1384 (1954).

(4) J. Tiomno: *Nuovo Cimento*, **6**, 255 (1957).

(5) R. E. Marshak: *Meson Physics* (New York, 1952), p. 143.

The matrix element which corresponds to the decay of π^0 into two γ-rays with four momenta K_1, K_2 and polarizations e_1 and e_2 respectively, through a baryon pair of mass M, is given (apart from trivial factors) by:

$$(3)\quad \mathcal{M}=\int \mathrm{d}^4p\ \mathrm{Sp}\left[\frac{i(p-k_2)-M}{(p-k_2)^2+M^2}(e_2+\alpha_1 e_2 k_2)\frac{ip-M}{p^2+M^2}(e_1+\alpha_1 e_1 k_1)\frac{i(p+k_1)-M}{(p+k_1)^2+M^2}\gamma_5\right],$$

with

$$(3')\quad \alpha_1=\frac{k}{2Mi}.$$

It is not difficult to show that:

$$(4)\quad \mathrm{Sp}\left[\{i(p-k_2)-M\}(e_2+\alpha_1 e_2 k_2)\{ip-M\}(e_1+\alpha_1 e_1 k_1)\{i(p+k_1)-M\}\gamma_5\right]=$$
$$-M(1+2iM\alpha_1-\alpha_1^2p^2-\alpha_1^2M^2)\,\mathrm{Sp}\,[e_2k_2e_1k_1\gamma_5]+$$
$$+i\alpha_1(p^2+M^2)[\mathrm{Sp}\,(e_2pe_1k_1\gamma_5)+\mathrm{Sp}\,(pe_2k_2e_1\gamma_5)].$$

This expression shows that the polarizations of the two photons are perpendicular to each other, so we can put:

$$(5)\qquad k_1=\omega(\gamma_3-\gamma_4),\qquad k_2=\omega(-\gamma_3-\gamma_4),\qquad e_1=\gamma_1,\qquad e_2=\gamma_2,$$

where ω is the photon frequency.

We get:

$$(6)\quad \mathcal{M}=2\int_0^1 x\,\mathrm{d}x\int_0^1 \mathrm{d}y\int \mathrm{d}^4p\ \frac{\{8M\omega^2(1+2iM\alpha_1-\alpha_1^2M^2)-8i\alpha_1\omega(p^2+M^2)p_\omega-8\alpha_1^2M\omega^2p^2\}}{[p^2+2px(k_1-k_1y-k_2y)+M^2]^3}.$$

Using:

$$(7)\qquad \int\frac{p^2\,\mathrm{d}^4p}{[(p-l)^2+a^2]^3}=i\pi^2D(a^2)-\frac{i\pi^2}{2}+l^2\frac{i\pi^2}{2a^2},$$

$$(8)\qquad \int\frac{p^2p_\lambda\,\mathrm{d}^4p}{[(p-l)^2+a^2]^3}=\frac{3}{2}i\pi^2l_\lambda D(a^2)+l^2l_\lambda\frac{i\pi^2}{2a^2}-\frac{5}{4}i\pi^2l_\lambda,$$

where:

$$(9)\qquad D(a^2)=\frac{1}{i\pi^2}\int\frac{\mathrm{d}^4p}{(p^2+a^2)^3}=\log\frac{\Lambda^2+a^2}{a^2}+\frac{a^2}{a^2+\Lambda^2}-1,$$

is a logaritmically divergent integral; Λ is a cut-off momentum and we obtain (*):

$$(10)\qquad \mathcal{M}=\frac{4i\pi^2\omega^2}{M}[1+iM\alpha_1+2i\alpha_1M-\alpha_1^2M^2)D(M^2)].$$

(*) Expressions (3), (4), (7) and (8) show that $\mathcal{M}$ is gauge invariant.

The usual expression for the lifetime of $\pi^0 \to 2\gamma$ decay through a proton-antiproton pair is given by:

$$\frac{1}{\tau} = \left(\frac{\alpha}{4\pi}\right)^2 \frac{\mu^3}{4\pi}\left(\frac{g_{\mathcal{N}\pi}}{M_{\mathcal{N}}}\right)^2 , \tag{11}$$

where μ is the meson mass and α the fine structure constant.

The formulas (1), (2) and (10) show that in order to obtain the corrected lifetime we must make the substitution:

$$\left(\frac{g_{\mathcal{N}\pi}}{M_{\mathcal{N}}}\right) \to \left\{\frac{g_{\mathcal{N}\pi}}{M_{\mathcal{N}}}\left[1 + \frac{k(p)}{2} + \left(k(p) + \frac{k^2(p)}{2} + \frac{k^2(n)}{2}\right) D(M^2_{\mathcal{N}})\right] + \right. \tag{12}$$

$$+ \frac{g_{\Sigma\pi}}{M_\Sigma}\left[\frac{k(\Sigma^+)}{2} - \frac{k(\Sigma^-)}{2} + \left(k(\Sigma^+) - k(\Sigma^-) + \frac{k^2(\Sigma^+)}{2} - \frac{k^2(\Sigma^-)}{2}\right) D(M^2_\Sigma)\right] -$$

$$\left. \frac{g_{\Xi\pi}}{M_\Xi}\left[1 + \frac{k(\Xi^-)}{2} + \left(k(\Xi^-) + \frac{k^2(\Xi^-)}{2} - \frac{k^2(\Xi^0)}{2}\right) D(M^2_\Xi)\right]\right\} .$$

Here we have neglected the mass difference between the components of each baryon multiplet.

Now for the pion-baryon interaction we shall take $g^2_{\mathcal{N}\pi} = g^2_{\Sigma\pi} = g^2_{\Xi\pi} = g^2_{\Lambda\pi}$ and the anomalous magnetic moments of the baryons will be given by (in a static approximation):

$$k(p) = -k(n) = k(\Sigma^+) = -k(\Sigma^-) = k(\Xi^0) = -k(\Xi^-) = 1.7 , \tag{13}$$

$$k(\Lambda^0) = k(\Sigma^0) = 0 ,$$

if we neglect the kaon contribution and the mass difference (Λ^0 and Σ with the same parity).

Using the experimental value $g^2_{\mathcal{N}\pi}/4\pi = 15$, we get for (11), $\tau \sim 4.5 \cdot 10^{-17}$ s.

Because of the inclusion of other baryons in the intermediate states and also their anomalous magnetic moments, the lifetime τ given by (11) will be modified according to the substitution (12). We can write:

$$\tau' = A\tau . \tag{14}$$

With the values (13) for the anomalous magnetic moments and the values of the masses $M_{\mathcal{N}} = 1\,837$, $M_\Sigma = 2\,335$, $M_\Xi = 2\,580$ in electron mass unit, we obtain the following set of values for A, according to the relative signs of $g_{\Sigma\pi}$ and $g_{\Xi\pi}$ to $g_{\mathcal{N}\pi}$:

$$\begin{cases} (a) & \dfrac{g_{\Sigma\pi}}{g_{\mathcal{N}\pi}} = \dfrac{g_{\Xi\pi}}{g_{\mathcal{N}\pi}} = 1 \to A \sim \dfrac{1}{15} , \\ (b) & \dfrac{g_{\Sigma\pi}}{g_{\mathcal{N}\pi}} = -\dfrac{g_{\Xi\pi}}{g_{\mathcal{N}\pi}} = 1 \to A \sim \dfrac{1}{14.5} , \\ (c) & \dfrac{g_{\Sigma\pi}}{g_{\mathcal{N}\pi}} = \dfrac{g_{\Xi\pi}}{g_{\mathcal{N}\pi}} = -1 \to A \sim 4 , \\ (d) & \dfrac{g_{\Sigma\pi}}{g_{\mathcal{N}\pi}} = -\dfrac{g_{\Xi\pi}}{g_{\mathcal{N}\pi}} = -1 \to A \sim 3.2 . \end{cases} \tag{15}$$

We used $D(M^2_{\mathcal{N}})$, $D(M^2_{\Sigma})$ and $D(M^2_{\Xi})$ given by (9) with $\Lambda = M_{\mathcal{N}}$. We have also considered $k(p) = -k(n) = 1.8$, $k(\Sigma^+) = -k(\Sigma^-) = 1.5$ and $k(\Xi^-) = -k(\Xi^0) = -1.3$ which follow from a static theory where we consider the experimental masses of the hyperons. The results (15) practically do not change.

The case (*a*) corresponds to Gell-Mann's model ([6]) while (*d*) to Tiomno's ([7]) one. For more generality we consider also the cases (*b*) and (*c*). In ref. ([4]) it was concluded that the lifetime of $\pi^0 \to 2\gamma$ for the possibilities (*a*) and (*d*) would increase by a factor of 10.) We see that this factor is reduced to 1/15 and 3 respectively, showing now a pronounced difference between Gell-Mann's and Tiomno's models, which is not apparent in the case where we do not consider anomalous magnetic moments of the baryons. We see then that if cases (*b*) and (*c*) do not hold, the measure of lifetime of $\pi^0 \to 2\gamma$ decay could decide between the two models in a clear way.

But to have more definite conclusions it should be necessary to have more experimental data about the pion-baryon interactions as well as the values of the anomalous magnetic moments.

* * *

We wish to thank the colleagues of our Institute and also Prof. J. Tiomno for interesting discussions.

([6]) M. Gell-Mann: *Phys. Rev.*, **106**, 1296 (1957).
([7]) J. Tiomno: *Nuovo Cimento*, **6**, 69 (1957).

L and M Conversion Ratios in ^{192}Os and ^{192}Pt.

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(ricevuto il 14 Dicembre 1959)

We have measured the internal conversion spectra of ^{192}Pt and ^{192}Os with a large (50 cm radius) iron-free double focusing spectrometer. In those cases where it was possible to separate L lines and no other lines were superposed on them, an excellent agreement was obtained with Sliv's [1] and Rose's [2] theoretical values.

Table I gives the results for the 205 keV transition, going from the first excited state into the ground state of even-even nuclei ^{192}Os. It should be pure $E2$. The agreement with theory is very good.

Table I. – *Theoretical and experimental $L_1 : L_2 : L_3$ ratios for $E2$ transition.*

γ-ray energy (keV)	Theory	Experiment
205.7	1 : 3,01 : 2,01	$1 : (2{,}9 \pm 0.1) : (2{,}0 \pm 0.1)$

For higher energy lines it was not possible to separate completely L_1 from L_2, but L_3 was separated. For that reason we compare $(L_1+L_2)/L_3$ ratios. Again the agreement is excellent as shown in Table II.

Table II. – *Theoretical and experimental $(L_1+L_2)/L_3$ ratios for $E2$ transition.*

γ-ray energy (keV)	Theory	Experiment
295.9	2.97	3.0 ∓ 0.2
316.4	3.23	3.2 ± 0.1
467.9	5.48	5.5 ± 0.3

[1] M. E. Rose: *Internal Conversion Coefficients* (Amsterdam, 1958).

[2] L. A. Sliv and I. M. Band: *Tablici koeficijentov unetrejnej konversii, Akad. Nauk SSSR* (1958).

The 316 keV transition goes from the first excited state of even-even ^{192}Pt and should be pure $E2$.

Table III shows that for two cases of mixed transitions L_2/L_1 and L_3/L_1 give the same percentage of $M1$ admixture. This indicates that precise measurements of L ratios can give reliable information on mixtures.

TABLE III.

γ-ray energy (keV)	$L_1 : L_2 : L_3$		Mixture	
	Theory $E2$	Experiment	L_2/L_1	L_3/L_1
136	1 : 8.9 : 6.7	1 : (3.22 $\pm$ 0.17) : (2.3 $\pm$ 0.1)	20% $M1$	20% $M1$
201	1 : 3.00 : 2.00	1 : (1.67 $\pm$ 0.05) : (1.07 $\pm$ 0.01)	16% $M1$	16% $M1$

The M lines were measured for 205, 295 and 316 keV. It was not possible to separate the lines completely, so that a graphical separation was made. The agreement with Rose's theoretical values is generally better than 10%. The results are given in the Table IV.

TABLE IV.

γ-ray energy (keV)	$M_1 : M_2$		$M_1 : M_3$		$(M_1 + M_2)/M_3$	
	Theory	Experiment	Theory	Experiment	Theory	Experiment
205.7	0.27	0.29 $\pm$ 0.03	0.38	0.44 $\pm$ 0.05	1.8	1.9 $\pm$ 0.1
295.9	0.46	0.49 $\pm$ 0.05	0.79	0.86 $\pm$ 0.04	2.5	2.7 $\pm$ 0.1
316.4	0.50	0.55 $\pm$ 0.01	0.93	1.06 $\pm$ 0.06	2.8	2.9 $\pm$ 0.2

A Fast Hyperfragment Produced by a Heavy Primary Cosmic Ray (*).

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(ricevuto il 21 Dicembre 1959)

The order of magnitude of the non-mesonic to mesonic ratio has been measured for several species of hyperfragments. With this ratio as well as the branching ratio of the mesonic decays and a determination of the lifetime, it is possible to find the partial decay rates of the various modes. This decay rate should throw light on the influence of nucleons on the strangeness violation in the decay. The effect should be greater for heavy hyperfragments ($A > 4$, $Z > 2$). The difficulty in attempts to measure lifetimes in emulsions is that the mean energy of the hyperfragments produced by K^--mesons and pions is so low that the moderation time is a very small fraction of the lifetime. Although the stopping power of bubble chambers is considerably less, the spatial resolution of these devices is such that the heavier hyperfragments can not be seen or identified. For these reasons it seems possible to measure lifetimes of heavy hyperfragments only in emulsions.

In an attempt to achieve higher nuclear excitation energies in interactions and correspondingly more energetic hyperfragments, a search has been made for hyperfragments produced by heavy primaries from the cosmic rays ($Z > 3$). A stack 6 in. × 12 in., G-5 600 μm, Ilford nuclear emulsion, which was flown over Texas for 8 hours above 104 000 ft. has been scanned for entering heavy primaries; these were followed until they either interacted or left the stack. A total of 141 interactions of heavy primaries ($Z \geqslant 3$) were found. All outgoing tracks from the interactions with ionization greater than about 4 times minimum were followed for 5 mm or until they stopped except for those tracks which were in the central cone,

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(**) On leave of absence from Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro.

and which were followed for 10 mm. Eight examples of secondary stars were found which were interpreted as interactions in flight of stable fragments. One event was found which is interpreted as a decay in flight of a $^4H_\Lambda$

A description of this event follows. The primary star was produced by a heavy primary, probably carbon. The track of the hyperfragment is nearly black (ionization corresponding to that of a proton of 18 MeV) and after 2.2 mm there is a change in direction of 63° and a reduction in ionization to that of a proton of 115 MeV. The angle between the directions of the heavy primary and the hyperfragment is 30°. The multiple scattering and gap measurements along the connecting track indicate a mass equal to 3600^{+1200}_{-500} MeV. This together with the marked decrease in ionization at the decay point, identify the connecting particle as a $^4H_\Lambda$. The characteristics of the hyperfragment are summarized in Table I.

TABLE I.

Characteristics of hyperfragment.

Length of hyperfragment track	2250 μm
Charge of connecting track	1 (by δ rays)
Mass measurement	3600^{+1200}_{-500}
Energy at time of decay	~ 70 MeV
Decay mode	$^4H_\Lambda \rightarrow p + 3n$
Proton energy from decay	115 MeV (96 MeV in c.m.s.)
Time lived	$0.37 \cdot 10^{-10}$ s

* * *

The authors are very grateful to Professor M. SCHEIN for his generous loan of the processed emulsion stack.

Risultati preliminari della determinazione dello spettro di bremsstrahlung dell'Elettrosincrotrone di Frascati.

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(ricevuto il 28 Dicembre 1959)

Sono dati i primi risultati sperimentali relativi alla determinazione dello spettro del fascio di bremsstrahlung dell'elettrosincrotrone di Frascati. Il fascio viene prodotto nel sincrotrone dal frenamento di elettroni, con energia $E_0 = 1000$ MeV da parte di un convertitore di tantalio. Gli elettroni, raggiunta tale energia, vengono fatti spiralizzare verso la parte interna della ciambella fino a che urtano il convertitore posto a 4 cm dall'orbita principale; l'ampiezza finale della radiofrequenza viene diminuita secondo un andamento che garantisce un tempo di collisione degli elettroni sul convertitore, di circa 2,5 ms. La distribuzione di intensità in funzione dell'angolo di emissione dei fotoni segue l'andamento previsto dalla formula di Schiff [1]. L'ampiezza angolare a mezzo massimo è di 3.5 mrad.

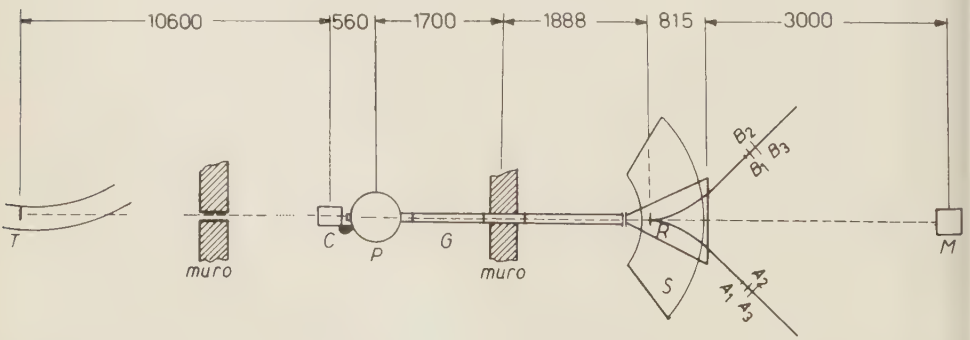

Fig. 1. – Disposizione sperimentale.

(1) L. I. Schiff: *Phys. Rev.*, **70**, 87 (1946).

La disposizione dell'esperienza è mostrata in Fig. 1 dove:

T è il convertitore di tantalio dell'elettrosincrotrone, il cui spessore è $s=0.13$ lunghezze di radiazione;

C un collimatore di piombo di lunghezza $l=30$ cm e con diametro $d=38$ mm; il semiangolo accettato è 1.8 mrad;

P un magnete pulitore, nel cui intraferro esiste un campo $B=15$ kG;

G una guida cilindrica in cui è praticato un vuoto di 0.1 torr;

S lo spettrometro a coppie la cui descrizione particolareggiata è data altrove [2];

R il convertitore dello spettrometro, costituito da un disco di alluminio del diametro di 6 cm e di spessore $\delta=1.08\cdot10^{-3}$ lunghezze di radiazione;

M un monitore costituito da un quantametro ad assorbimento totale;

$A_1A_2A_3$ e $B_1B_2B_3$ i telescopi di contatori a scintillazione, posti in posizioni tali da rivelare solo le coppie simmetriche di elettroni [3].

La selezione in energia delle coppie viene effettuata variando il valore del campo magnetico, nell'intraferro dello spettrometro, da 1 a 10 kG. La precisione con cui è determinata l'energia delle traiettorie è di $\pm3‰$. L'intervallo percentuale di energia accettata dai telescopi è $\Delta E/E=2.7\%$. Le uscite dei contatori $A_1A_2A_3$ del telescopio A e dei contatori $B_1B_2B_3$ del telescopio B sono collegate rispettivamente a due circuiti di coincidenze triple con un tempo di risoluzione $\tau=6$ ns.

Le uscite delle due coincidenze triple sono inviate ad una coincidenza doppia simultanea e ad una coincidenza doppia ritardata con tempi di risoluzione $\tau=10$ ns. Si ha così la possibilità di misurare il numero di coincidenze dovuto a coppie simmetriche di elettroni e contemporaneamente il numero di coincidenze spurie dovuto a coppie asimmetriche e quelle accidentali dovute al fondo.

Il quantametro ad assorbimento totale, del tipo descritto da R. R. Wilson [4], dà un'informazione integrale dell'energia trasportata dal fascio rivelando la carica elettrica

$$Q = q\int_0^{E_0} KN(K)\,\mathrm{d}K\,,$$

corrispondente a Q/q MeV di energia irraggiata, essendo $q=2.07\cdot10^{-19}$ C/MeV una costante caratteristica dello strumento e $N(K)\,\mathrm{d}K$ il numero di fotoni di energia compresa fra K e $K+\mathrm{d}K$.

Nel seguito si danno i risultati sperimentali della funzione « intensità della radiazione di bremsstrahlung » definita da

$$I\left(\frac{K}{E_0}\right) = KN(K)\,,$$

dove E_0 è l'energia massima degli elettroni.

[2] G. Bologna, G. Diambrini, A. S. Figuera, U. Pellegrini, B. Rispoli, A. Serra e R. Toschi: Nota interna n. 17 dei Laboratori Nazionali di Frascati (28 Settembre 1959).

[3] Le posizioni dei contatori sono state determinate mediante misure fatte con la tecnica del filo e controllate mediante il calcolo delle traiettorie.

[4] R. R. Wilson: *Nucl. Instr.*, **1**, 101 (1957).

Si può ora calcolare che il rapporto tra le intensità relative a due energie K_0, K_n dei fotoni è dato dalla espressione

$$\frac{I_n(K_n/E_0)}{I_0(K_0/E_0)} = \frac{Q_0}{Q_n} \frac{N_n - (S_n + F_n)}{N_0 - (S_0 + F_0)} \frac{\sigma_p(K_0, \frac{1}{2}) + \sigma_t(K_0, \frac{1}{2})}{\sigma_p(K_n, \frac{1}{2}) + \sigma_t(K_n, \frac{1}{2})},$$

dove N_n, N_0 rappresentano il numero di coincidenze contate alle energie K_n e K_0; S_n, S_0 sono le coincidenze dovute a coppie spurie; F_n, F_0 sono le coincidenze dovute al fondo, $N_n - (S_n + F_n)$ e $N_0 - (S_0 + F_0)$ rappresentano quindi il numero di coppie simmetriche prodotte rispettivamente da fotoni di energia K_n e K_0; $\sigma_p(K_0, \frac{1}{2})$, $\sigma_t(K_0, \frac{1}{2})$, e $\sigma_p(K_n, \frac{1}{2})$, $\sigma_t(K_n, \frac{1}{2})$ sono le sezioni d'urto di produzione di coppie [5] nel campo del nucleo e nel campo dell'elettrone, per coppie simmetriche e per fotoni di energia K_0 e K_n; Q_0, Q_n sono le cariche liberate nel quantametro dal fascio γ [6].

Si procede così alla misura del rapporto I_n/I_0 in funzione della energia K_n dei fotoni. La dose scelta per ogni singolo conteggio è di $\sim 1.4 \cdot 10^{10}$ quanti-equivalenti corrispondenti a $3 \cdot 10^{-6}$ C.

Ogni conteggio è stato ripetuto da 5 a 10 volte. Per minimizzare le coincidenze spurie si è mantenuta l'intensità del fascio non superiore a circa 10^{10} quanti equivalenti/minuto dopo la collimazione. In queste condizioni sperimentali la riproducibilità del numero di conteggi è dello stesso ordine di grandezza dell'errore statistico percentuale:

$$\frac{1}{\sqrt{N}} \leq 1\% .$$

I dati sperimentali sono riportati nella Fig. 2. La intensità di bremsstrahlung

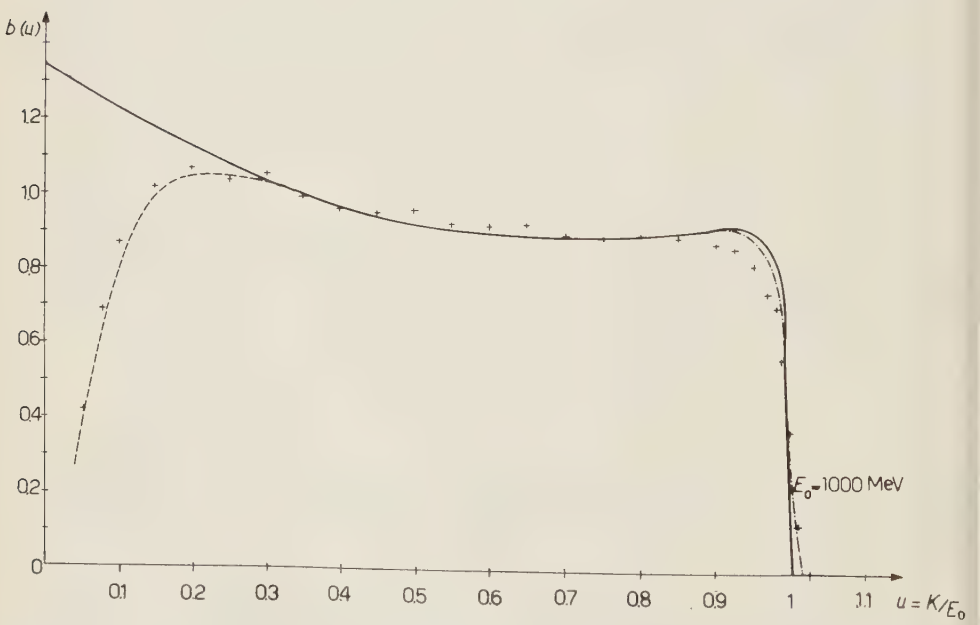

Fig. 2. – Spettro di bremsstrahlung. —— Teorico (Bethe e Heitler + Wheeler e Lamb). - - - Teorico corretto per lo scattering nel convertitore dello spettrometro e per le dimensioni verticali finite dei contatori. –·–·– Teorico corretto per il potere risolutivo finito dei contatori. + Dati sperimentali.

misurata a 700 MeV è stata fatta coincidere con il punto della curva teorica normalizzata calcolata per $E_0 = 1\,000$ MeV.

In Fig. 2 è riportata (curva a tratto pieno) la espressione

$$b(u) = \frac{K(\sigma_n + \sigma_{el})}{\int_0^1 K(\sigma_n - \sigma_{el})\,du},$$

in funzione di $u = K/E_0$, dove $\sigma_n(K, u)\,dK$, $\sigma_{el}(K, u)\,dK$ sono le sezioni d'urto di bremsstrahlung calcolate da Bethe e Heitler, e Wheeler e Lamb nei lavori citati [5].

La curva tratteggiata che si discosta da quella a tratto pieno nella regione delle basse energie dei fotoni, è ottenuta sottraendo alla curva teorica di bremsstrahlung le perdite di conteggio dovute allo scattering degli elettroni nel convertitore di alluminio e alla dimensione verticale finita degli scintillatori plastici [7]. La curva a punti e linee in prossimità dell'energia massima rappresenta la correzione dovuta al potere risolutivo finito in energia dei contatori ($\Delta E/E \sim 2{,}7\%$).

Come si vede dalla Fig. 2 l'andamento della curva sperimentale si discosta da quello teorico relativo ad un convertitore infinitamente sottile alle alte energie; più precisamente si osserva che per $0{,}85 \leqslant u \leqslant 1$, i punti sperimentali sono tutti al di sotto della curva teorica. Tale effetto può essere attribuito alle seguenti cause:

1) Gli elettroni che urtano il convertitore per un tempo di 2,5 ms hanno una indefinizione in energia del 0,8%. Inoltre una piccola percentuale di elettroni può investire il convertitore prima di aver raggiunto l'energia finale di 1 000 MeV.

2) Lo spessore finito del convertitore γ produce una ulteriore indefinizione in energia degli elettroni prima dell'irraggiamento.

3) La collimazione tende a diminuire lo spessore efficace del convertitore.

Misure e calcoli per valutare il peso delle varie cause sull'andamento sperimentale dello spettro sono in corso.

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[5] H. Bethe e W. Heitler: *Proc. Roy. Soc.*, A **146**, 83 (1934); J. A. Wheeler e W. E. Lamb: *Phys. Rev.*, **55**, 858 (1939); H. Davies, H. A. Bethe e L. C. Maximon: *Phys. Rev.*, **93**, 788 (1954).

[6] Le misure sono state generalmente riferite alla stessa dose cosicchè risulta $Q_0 = Q_n$.

[7] Non si è tenuto conto della perdita di conteggio dovuta all'angolo di emissione delle coppie di elettroni.

Note on the Effect of the Nuclear Energy Gap on the Optical Model Potential.

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(ricevuto il 28 Dicembre 1959)

Recently some effects of the strong nucleon-nucleon correlations in nuclear matter on the optical model potential have been examined using the Watson theory and Jastrow type wave functions [1]. The correlation wave functions introduce the non-degeneracy effect of the Fermi gas espressed both in the exclusion principle operator in the t-matrix of the scattering pair and in the nuclear wave function.

On the other hand, the Bardeen, Cooper and Schrieffer theory of superfluidity of systems of fermions has been extended to nuclear matter and finite nuclei by a number of authors [2-6]. In particular, the *BCS* type pairing forces seem to be responsible for the pairing energies and the well known fact of the energy gap in even-even nuclei. The aim of the present note is to give an estimate of the effect of the pairing correlation on the optical model potential.

Recentrly Gor'kov [7] presented a Green's function type of derivation of the results of the *BCS* theory and Migdal [6] extended this phenomenological model to arbitrary systems of quasiparticles with the particular application to finite nuclei. He also presented a calculation of the nuclear moments of inertia. Migdal's method seems to be more straightforward and more practical than the canonical transformation method of Belyaev [5] for computing the intrinsic properties of interacting systems of fermions with the energy gap. It allows calculation of any higher order corrections introduced by perturbation interactions of the type of the impurities in superconductors. A number of applications to some intrinsic properties of finite nuclei is the subject of a future publication.

The essential point of Migdal's method is the use of the single particle mixed

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[1] J. Dabrowski and J. Sawicki: *Nucl. Phys.*, **13**, 621 (1959).
[2] A. Bohr, B. R. Mottelson and D. Pines: *Phys. Rev.*, **110**, 936 (1958).
[3] V. G. Soloviev: *Nucl. Phys.*, **9**, 655 (1959).
[4] V. G. Soloviev: *Nuovo Cimento*, **10**, 1022 (1958).
[5] S. T. Belyaev: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **31**, no. 11 (1958).
[6] A. B. Migdal: *Žurn. Eksp. Teor. Fiz.*, **37**, 249 (1959).
[7] L. P. Gor'kov: *Žurn. Eksp. Teor. Fiz.*, **34**, 735 (1958).

density matrix related to the single particle Green's function as:

$$\varrho(\boldsymbol{r}, \boldsymbol{r}') = -iG(\boldsymbol{r}, \boldsymbol{r}', \tau = t - t')|_{\tau \to -0} = \frac{1}{2\pi i}\int_C G(\boldsymbol{r}, \boldsymbol{r}'\varepsilon)\,\mathrm{d}\varepsilon\,, \tag{1}$$

where the contour of integration C is the real axis plus upper half plane semicircle. The Green's function refers to a system with a δ-function type weak pairing force and characterized by an energy gap Δ. Using the usual prescription for defining the poles one finds in the simplest case of no additional perturbation force present:

$$\varrho(\boldsymbol{r}, \boldsymbol{r}') = \sum_{\nu,\nu'} \varrho_{\nu\nu'}\varphi_\nu(\boldsymbol{r})\varphi^*_{\nu'}(\boldsymbol{r}')\,, \qquad \varrho_{\nu\nu'} = \frac{1}{2}\left(1 - \frac{\varepsilon_\nu - \lambda}{\sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2}}\right)\delta_{\nu\nu'}\,, \tag{2}$$

where ε_ν's and φ_ν's are the eigenvalues and eigenfunctions of a selfconsistent single particle hamiltonian, λ is the chemical potential, and Δ is the constant gap. $\varrho_{\nu\nu}$ of eq. (2) correspond to the squared coefficient of the canonical trasformation of BELYAEV [5] $= V_\nu^2$. If the number of particles in the system integracting through the pairing force is N, then

$$N = \sum_\nu \frac{1}{2}\left(1 - \frac{\varepsilon_\nu - \lambda}{\sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2}}\right), \tag{3}$$

which is the equation to determine the chemical potential λ.

Using Watson's theory and dropping the cluster terms one can write the generally nonlocal optical model potential in momentum space in the form:

$$\langle \boldsymbol{k}_0' | V | \boldsymbol{k}_0 \rangle = \int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_1' \langle \boldsymbol{k}_{01}' | \tilde{t} | \boldsymbol{k}_{01} \rangle\, \delta(\boldsymbol{K}_{01} - \boldsymbol{K}_{01}')(2\pi)^{-3} \cdot \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\, \exp\left[i(\boldsymbol{k}_1\boldsymbol{r} - \boldsymbol{k}_1'\boldsymbol{r}')\right]\varrho(\boldsymbol{r}, \boldsymbol{r}')\,, \tag{4}$$

where $\tilde{t}$ is the effective antisymmetrized t-matrix of the interacting pair; $\boldsymbol{k}_0$, $\boldsymbol{k}_0'$, and $\boldsymbol{k}_1$, $\boldsymbol{k}_1'$ are the initial and final momenta of the projectile, and of the target particle, respectively; $\boldsymbol{k}_{01}$ and $\boldsymbol{k}_{01}'$ are the relative — and $\boldsymbol{K}_{01}$ and $\boldsymbol{K}_{01}'$ — the total momenta of the respective states. In the case of infinite medium $\varepsilon_\varkappa = \hbar^2\varkappa^2/2M$ takes on the role of ε_ν and $\varrho(\boldsymbol{r}, \boldsymbol{r}')$ is a function of $|\boldsymbol{r} - \boldsymbol{r}'|$ only. We find:

$$\varrho(\boldsymbol{r} - \boldsymbol{r}') = 4(2\pi)^{-3}\int \varrho(\varkappa)\,\exp\left[i\boldsymbol{\varkappa}(\boldsymbol{r} - \boldsymbol{r}')\right]\mathrm{d}\boldsymbol{\varkappa}\,, \tag{5}$$

$$\varrho(\varkappa) = \frac{1}{2}\left(1 - \frac{\varepsilon_\varkappa - \lambda}{\sqrt{(\varepsilon_\varkappa - \lambda)^2 + \Delta^2}}\right). \tag{5a}$$

The factor 4 stems from the assumed degeneracy in spin and isotopic spin, *i.e.*, the same energy gap for all the spin states and for neutrons and protons. This is a crude assumption to give an idea of the effect. The chemical potential is to be

determined from the equation:

$$\varrho_0 = \frac{2k_F^3}{3\pi^2} = \frac{2}{\pi^2}\int_0^\infty \varkappa^2 \,\mathrm{d}\varkappa \varrho(\varkappa) \,. \tag{6}$$

The optical potential now becomes:

$$\langle \boldsymbol{k}_0' | V | \boldsymbol{k}_0 \rangle = \delta(\boldsymbol{k}_0 - \boldsymbol{k}_0') V(k_0) \,, \tag{7}$$

$$V(k_0) = 4\int \mathrm{d}\boldsymbol{\varkappa} \tilde{t}\left(\frac{\boldsymbol{k}_0 - \boldsymbol{\varkappa}}{2}, \boldsymbol{k}_0 + \boldsymbol{\varkappa}\right) \varrho(\varkappa) \,, \tag{7a}$$

Here $\tilde{t}$ is the on-energy-shell diagonal $\tilde{t}$-matrix. The dependence of $V(k_0)$ on Δ is two-fold: *a*) the exclusion principle operator $Q(k_0', k_1')$ involved in the intermediate states occuring in the integral equation for the t-matrix has to be replaced by $\tilde{Q}(k_0', k_1') = (1 - \varrho(k_0'))(1 - \varrho(k_1'))$; *b*) the density factor $\varrho(\varkappa)$ in eq. (7*a*). The effect *a*), *i.e.*, that of $\tilde{Q} - Q$ on the $\tilde{t}$-matrix, was examined for the particular numerical case discussed below and found to be negligible. It turns out that for the YAMAGOUCHI separable potential discussed in [8] the effect of the Q-operator on the imaginary part of the optical potential V_I is not very important (it reduces V_I to 75% of its value with no Q present for $k = 1.5k_F$, 89% for $k = 2k_F$, and 91% for $k = 3k_F$). The effect of Q on the real part of the potential is more important for that particular case at medium energies. In order to estimate the effect *b*) Yamagouchi $\tilde{t}$-matrix is used and the Q-operator, and consequently the total momentum dependence are suppresed. In the absence of Q we can write:

$$V(k_0) = \frac{16\pi}{\nu} k_F^3 \int_0^\infty \mathrm{d}x x \tilde{t}(k_F x)\left[4\nu x - \frac{1}{2}\sqrt{[(\nu + 2x)^2 - \lambda/\varepsilon_F]^2 + (\Delta/\varepsilon_F)^2} + \right. \tag{8}$$

$$\left. + \frac{1}{2}\sqrt{[(\nu - 2x)^2 - \lambda/\varepsilon_F]^2 + (\Delta/\varepsilon_F)^2}\right] ,$$

where $\nu = k_0/k_F$, $x = k/k_F$, $\varepsilon_F = \hbar^2 k_F/2M$. In the case of no energy gap present eq. (8) reduces to the familiar form:

$$V_0(k_0) = \frac{16\pi}{\nu} k_F^3 \int_{\frac{1}{2}(\nu-1)}^{\frac{1}{2}(\nu+1)} \mathrm{d}x x \tilde{t}(k_F x)[1 - \nu^2 + 4\nu x - 4x^2] \,. \tag{9}$$

The average energy was taken for a heavy nucleus (^{238}U) from MIGDAL [6] as $\Delta = (A - Z/A)\Delta_n + (Z/A)\Delta_p \approx 0.73$ MeV. λ is to be computed from the equation

$$1 = \frac{3}{2}\int_0^\infty \mathrm{d}x x^2 \left(1 - \frac{(x^2 - \lambda/\varepsilon_F)}{\sqrt{(x^2 - \lambda/\varepsilon_F)^2 + (\Delta/\varepsilon_F)^2}}\right) . \tag{10}$$

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[8] L. VERLET and J. GAVORET: *Nuovo Cimento*, **10**, 505 (1958).

As a results of the smallness of $(\Delta/\varepsilon_F)^2$ the value of λ is very close to ε_F, particularly for a high density, *i.e.*, a large k_F. The low density case was taken ($k_F = 1.25f^{-1}$) as for higher densities ($k_F \sim 1.4f^{-1}$) the effect is entirely negligible. It turns out that for $k_0 = 1.5k_F$ the difference between Im $V(k_0)$ of Eq. (8) and Im $V_0(k_0)$ of eq. (9) is less than 1% and, in fact, negligible. A similar situation holds for the real part and for higher energies. This result is essentially independent of the particular choice of the potential and originates from the smallness of the quantity $(\Delta/\varepsilon_F)^2$.

We conclude that the effect of the energy gap on the optical potential in the case of nuclear matter is negligible. In the case of finite nuclei there may be effects associated with the nuclear surface (*i.e.*, the derivatives of the density) such as the spin-orbit coupling for which the energy gap may be important. In the case of spherical nuclei the single particle density is of the form:

$$\varrho(r) = \sum_{nl} \frac{1}{2}\left(1 - \frac{\varepsilon_{nl} - \lambda}{\sqrt{(\varepsilon_{nl} - \lambda)^2 + \Delta^2}}\right) \frac{2l+1}{4\pi} [N_{nl} R_{nl}(r)]^2 , \tag{11}$$

where $N_{nl}R_{nl}(r)$ is a normalized radial shell model wave function. In the case of spheroidal nuclei characterized by Nilsson's « intrinsic » functions [9] we have:

$$\varrho(\boldsymbol{r}') = \sum_{N\Omega a} \frac{1}{2}\left(1 - \frac{\varepsilon_{N\Omega a} - \lambda}{\sqrt{(\varepsilon_{N\Omega a} - \lambda)^2 + \Delta^2}}\right) |X_{N\Omega a}(\boldsymbol{r}')|^2 , \tag{12}$$

where:

$$|X_{N\Omega a}(\boldsymbol{r}')|^2 = \sum_{l,l',\Lambda=\Omega\pm\frac{1}{2}} a_{l\Lambda} a_{l'\Lambda} R_{Nl}(r') R_{Nl'}(r') Y_{l\Lambda}(\Omega') Y^*_{l'\Lambda}(\Omega') . \tag{12a}$$

The spin-orbit doublet splittings can be calculated using the method of [10,11] and in particular from eqs. (8) and (14) of [11]. Such calculations are in preparation.

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Several stimulating conversations with Dr. N. K. Glendenning are gratefully acknowledged.

[9] S. G. Nilsson: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **29**, no. 16 (1955).

[10] J. Sawicki and R. Folk: *Nucl. Phys.*, **11**, 368 (1959); in addition to the errata to this paper given in [11] a factor 4 is missing in Eqs. (A-II.1) and (A-II.6) of Appendix **2**.

[11] J. Sawicki: *Nucl. Phys.*, **13**, 350 (1959).

An Estimate of the Cross Section for Production of Positive K-Mesons by Electrons (*).

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(ricevuto il 18 Gennaio 1960)

The purpose of this note is to discuss the possibility of using the electron beam of the Frascati synchrotron to study the reaction

$$e^- + p \to e^- + \Lambda^0 + K^+ . \tag{1}$$

The threshold energy for this reaction is about 910 MeV whereas the beam energy may reach 1050 MeV. The cross-section for electroproduction of K^+ may be related to that for photoproduction by means of a formulation of the method of Weizsäcker-Williams given by Dalitz and Yennie (1) in their work on electroproduction of pions. It must be stressed that our results are less reliable than those obtained by Dalitz and Yennie, since the threshold for photoproduction of K-mesons (about 910 MeV) is much larger than that for photoproduction of π-mesons. The electron, on producing a K-meson by its virtual photon field, may undergo large angle scattering. (For threshold production the final electron energy is only 140 MeV). Therefore it is not correct to treat the Fourier components of the electromagnetic field of the incident electron as incoherent and the virtual photons as transversal, as is done in the W.-W. method. Dalitz and Yennie have shown that this difficulty can be overcome by taking into account the effect of longitudinal virtual photons. Their standard formula

$$N_e^{\mathrm{st}}(p, k_f)\frac{\mathrm{d}k_f}{k_f} = \frac{\alpha}{\pi}\frac{\mathrm{d}k_f}{k_f}\left\{\frac{\varepsilon^2 + \varepsilon'^2}{p^2}\log\frac{\varepsilon\varepsilon' + pp' + m^2}{m(\varepsilon - \varepsilon')} - \frac{(\varepsilon + \varepsilon')^2}{2p^2}\log\frac{p + p'}{p - p'} - \frac{p'}{p}\right\},$$

for the number of virtual photons may perhaps be employed also in the case of reaction (1). In fact, using eq. (1.5a) of ref. (1) we have estimated that 53.4% of the virtual photons of 911 MeV energy, 52.1% of those of 920 MeV, 44.9% of those of 971 MeV, 36.6% of those of 996.5 MeV would correspond to electrons with directions

(*) Presented in partial fulfilment of the requirements for the degree of Dottore in Fisica (Tesina).

(1) R. Dalitz and D. R. Yennie: *Phys. Rev.*, **105**, 1598 (1957).

of motion forming angles of less than 6° with the direction of the incident beam, whereas 65.2% of the virtual photons of 911 MeV energy, 63.9% of those of 920 MeV, 61.4% of those of 945 MeV, 58.1% of those of 971 MeV, 51.4% of those of 996.5 MeV would correspond to electrons with directions of motion forming angles of less than 12° (energy of the beam about 1022 MeV).

A perturbation calculation of the cross-section for photoproduction of K-mesons has been performed by KAWAGUCHI and MORAVCSIK [2] to the first order in the coupling constant. A comparison of their results (for the case of heavy particles with no anomalous magnetic moments and assuming pseudoscalar coupling) with the experimental data of DONOHO and WALKER [3] has shown that the coupling constant is $(G/4\pi)^2 \sim 2.9$. Using this value in the formula

$$\Phi_{\text{el. tot.}} = \int_{k_f=910}^{k_f=1022} N_e^{\text{st}}(p, k_f)\, \Phi_{\text{ph. tot.}}(k_f)\, \frac{\mathrm{d}k_f}{k_f},$$

for the total cross section for our reaction (1), we find $3.65 \cdot 10^{-33}$ cm^2 for $p = 1022$ MeV.

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[2] M. KAWAGUCHI and M. J. MORAVCSIK: *Phys. Rev.*, **107**, 563 (1957).
[3] P. L. DONOHO and R. L. WALKER: *Phys. Rev.*, **112**, 981 (1958).

LIBRI RICEVUTI E RECENSIONI

F. H. Mitchell – *Fundamentals of Electronics* (Addison-Wesley Publishing Co. Inc., Reading Mass., USA; London, England); prezzo $ 6,50; II ed., Settembre 1959, pag. XI+260.

Secondo quanto afferma l'autore stesso, il libro è stato concepito con una doppia finalità: 1) costituire le fondamenta della scienza elettronica nei giovani i quali intendono costruire su queste fondamenta, vale a dire per i futuri specialisti; 2) tentare di dare a coloro che non intendono divenire degli elettronici, una visione generale ed una sensibilità dei fenomeni che si svolgono nelle apparecchiature elettroniche che oggi qualsiasi fisico è costretto ad usare.

Il libro è quindi dedicato a giovani studenti di « undergraduate courses »; è senza dubbio al di sotto del livello dei nostri corsi universitari del 1° biennio sebbene sia più specializzato, ma potrebbe costituire un ottimo testo per i corsi superiori dei nostri istituti tecnici industriali.

Nel I e II capitolo si introducono le nozioni elementari riguardanti circuiti in corrente continua ed alternata, i componenti e le leggi fondamentali.

Nel cap. III si introduce in forma elementare la emissione termoionica e nei successivi capitoli IV, V, VI, l'autore tratta delle valvole termoioniche.

Nei cap. VII, VIII si tratta, in forma del tutto convenzionale, degli amplificatori per onde sinusoidali; non si accenna agli amplificatori in regime impulsivo.

Agli oscillatori sinusoidali è dedicato il cap. IX i successivi capitoli sono dedicati alle valvole a gas ed alle valvole speciali.

Un capitolo è dedicato ai circuiti non lineari, quali limitatori, multivibratori, sempre visti dal punto di vista di segnali periodici.

Chiude il libro un capitolo sui voltmetri elettronici e gli oscillografi a raggi catodici.

Ciascun capitolo è corredato di numerosi esercizi da svolgere.

I vari argomenti sono trattati con estrema semplicità; ogni rigore è stato volutamente abbandonato. Nei limiti indicati dall'autore possiamo concludere che può considerarsi un libro ben riuscito.

Brunelli Rispoli

W. Kaplan – *Ordinary differential equations*. Addison-Wesley Publishing Company, Inc. Reading, Mass., U.S.A., 1958, pagine 534.

Questo volume, di carattere didattico, è diviso in 12 Capitoli. Nei primi otto è esposta la teoria elementare delle equazioni differenziali ordinarie, così come si svolge nel nostro biennio propedeutico.

È da rilevare la ricchezza di esempi e di applicazioni a problemi di meccanica e di fisica e pure notevoli sono le collezioni di esercizi posti alla fine di ogni capitolo, con indicazione delle relative soluzioni.

Più interessanti e di carattere più elevato sono il Cap. 9 dedicato alla

ricerca di soluzioni sotto forma di serie di potenze, il Cap. 10 sui metodi di calcolo numerico delle soluzioni, il Cap. 11 rivolto all'analisi delle soluzioni nel piano delle fasi ed infine il Cap. 12 ove sono dati i teoremi di esistenza e di unicità. Anche in questi ultimi Capitoli tutti i concetti ed i teoremi sono ben illustrati da esempi, figure, esercizi di notevole interesse.

In complesso il libro può essere giudicato ben riuscito e di indubbia utilità per gli studenti di fisica o di ingegneria.

A. GHIZZETTI

MAHLON M. DAY – *Normed linear spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 21. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958, pag. 139.

Questo libro contiene una succinta introduzione allo studio degli spazi lineari normali; vi si parla anche degli spazi lineari topologici, ma limitatamente a quanto serve per lo studio predetto. Tutta l'esposizione ha carattere astratto e non vi è alcun cenno a possibili applicazioni; il libro appare pertanto dedicato essenzialmente ai cultori ed ai ricercatori di matematica pura.

Nel Cap. I sono introdotti gli spazi lineari e gli spazi lineari topologici e nel Cap. II gli spazi lineari normali. Seguono altri cinque Capitoli ove si tratta di vari importanti concetti relativi ai predetti spazi (completezza, compattezza, riflessività; convergenza assoluta, basi; insiemi compatti e convessi, spazi di funzioni continue; reticoli vettoriali; geometria metrica negli spazi normali). Un ultimo breve Capitolo, accompagnato da una bibliografia, è destinato a servire di guida al lettore che voglia approfondire lo studio di certi argomenti appena accennati nel libro o addirittura ignorati.

A. GHIZZETTI

J. BASS – *Cours de Mathématiques*. Ed. Masson et C.ie, Paris, 1956; pag. 916.

Si tratta di un grosso volume ove sono esposti in 41 Capitoli e 2 Appendici tutti quegli argomenti classici di Analisi Matematica che, di solito, formano oggetto dei nostri corsi del primo biennio e di un successivo corso di Analisi Superiore. Il libro è scritto per coloro che si dedicano alle applicazioni della matematica, non per matematici puri, giacchè in esso tutte le nozioni sono presentate secondo gli schemi tradizionali, senza considerazione nè di questioni critiche, nè dei moderni indirizzi astratti ed assiomatici.

Ecco un succinto elenco degli argomenti trattati: algebra lineare, calcolo differenziale ed integrale, sviluppi in serie, geometria differenziale delle curve e delle superficie, funzioni analitiche, equazioni differenziali ordinarie o a derivate parziali, calcolo delle variazioni, nomografia. Tutti questi argomenti sono trattati abbastanza a fondo con ricchezza di esempi e di applicazioni.

A. GHIZZETTI

R. P. BOAS jr. and R. CREIGHTON BUCK – *Polynomial expansions of analytic functions*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 19. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958, pag. 77.

Lo scopo di questo volumetto è di descrivere e di applicare un metodo generale per sviluppare una funzione analitica $f(z)$ in serie di appropriate funzioni $p_n(z)$ (in particolare di polinomi):

$$(1) \qquad f(z) = \sum c_n p_n(z) .$$

Tale metodo è il seguente: assegnate le $p_n(z)$ si cerca di determinare delle fun-

zioni $u_n(w)$ in modo che, introdotto il *nucleo* $K(z, w) = \sum p_n(z)\, u_n(w)$, sia possibile rappresentare la $f(z)$ con una formula del tipo

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} K(z, w)\, F(w)\, dw ,$$

con F funzione opportuna (dipendente da f). È chiaro che, fatto ciò, i coefficienti c_n dello sviluppo (1) hanno necessariamente l'espressione

$$c_n = \frac{1}{2\pi i} \int_{\Gamma} u_n(w) F(w)\, dw .$$

Osservato che questi c_n sono dei funzionali $L_n(f)$ della funzione f, viene anche considerato il problema inverso (detto di interpolazione): esprimere la f per mezzo di uno sviluppo del tipo (1) supponendo assegnata una successione di opportuni funzionali $c_n = L_n(f)$.

Il metodo di cui sopra è analizzato nel Cap. I e successivamente applicato a molti casi particolari che riguardano le funzioni intere (Cap. II) e le funzioni regolari nell'origine (Cap. III). Infine il Cap. IV tratta di notevoli applicazioni a teoremi di unicità ed equazioni funzionali.

Il libro può essere utile anche nello studio di problemi applicativi.

A. GHIZZETTI

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Questo Fascicolo è stato licenziato dai torchi il 28-II-1960